

Stability Conditions on A_n -Singularities

Akira Ishii, Kazushi Ueda, and Hokuto Uehara

Abstract

We study the spaces of locally finite stability conditions on the derived categories of coherent sheaves on the minimal resolutions of A_n -singularities supported at the exceptional sets. Our main theorem is that they are connected and simply-connected. The proof is based on the study of spherical objects in [30] and the homological mirror symmetry for A_n -singularities.

1 Introduction

The theory of stability conditions on triangulated categories is introduced by Bridgeland [9] based on the work of Douglas et al. [2, 15, 16, 17, 18] on the stability of BPS D-branes. It is a fine mixture of the theory of t -structures [3] and the slope stability [36], which allows us to represent any object in a triangulated category as a successive mapping cone of semistable objects in a unique way. He proved that the set $\text{Stab } \mathcal{T}$ of stability conditions on a triangulated category \mathcal{T} satisfying an additional assumption called local-finiteness has a natural structure of a complex manifold, and proposed to study this manifold as an invariant of \mathcal{T} . Since the definition of stability conditions uses only the triangulated structure of \mathcal{T} , the group $\text{Auteq } \mathcal{T}$ of triangle autoequivalences of \mathcal{T} naturally acts on the manifold $\text{Stab } \mathcal{T}$, suggesting a geometric approach to study the structure of $\text{Auteq } \mathcal{T}$.

Such an approach has been pursued by Bridgeland himself [11] when \mathcal{T} is the bounded derived category $D^b \text{coh } X$ of coherent sheaves on a complex algebraic K3 surface X , leading him to the following remarkable result and conjecture: There is a distinguished connected component $\Sigma(X)$ of the space $\text{Stab } X$ of locally finite, numerical stability conditions on $D^b \text{coh } X$. His conjecture is:

- (i) $\Sigma(X)$ is preserved by $\text{Auteq } D^b \text{coh } X$, and
- (ii) $\Sigma(X)$ is simply-connected.

Assuming this conjecture, he could prove that $\text{Auteq } D^b \text{coh } X$ is an extension of the index two subgroup $\text{Aut}^+ H^*(X, \mathbb{Z})$ of the group of Hodge isometries of the Mukai lattice of X by the fundamental group $\pi_1 \mathcal{P}_0^+(X)$ of the period domain of X .

For a positive integer n , let

$$f : X \rightarrow Y = \operatorname{Spec} \mathbb{C}[x, y, z]/(xy + z^{n+1})$$

be the minimal resolution of the A_n -singularity. Let further \mathcal{D} be the bounded derived category $D^b \operatorname{coh}_Z X$ of coherent sheaves on X supported at the exceptional set Z , and \mathcal{C} be its full triangulated subcategory consisting of objects E satisfying $\mathbb{R}f_* E = 0$. The categories \mathcal{C} and \mathcal{D} serve as toy models of the derived categories of coherent sheaves on $K3$ surfaces. The main result in this paper is the following:

Theorem 1. *$\operatorname{Stab} \mathcal{C}$ and $\operatorname{Stab} \mathcal{D}$ are connected, and $\operatorname{Stab} \mathcal{D}$ is simply-connected.*

This result, together with the simply-connectedness of a distinguished connected component of $\operatorname{Stab} \mathcal{C}$ proved by Thomas [42], shows that the above conjecture of Bridgeland holds in these cases. When $n = 1$, the connectedness of $\operatorname{Stab} \mathcal{D}$ has also been proved by Okada [37].

The basic strategy of our proof for the connectedness of $\operatorname{Stab} \mathcal{D}$ is to find a stability condition such that structure sheaves of all the closed points are stable in any given connected component of $\operatorname{Stab} \mathcal{D}$. Since the set of such stability conditions form a distinguished connected open subset of $\operatorname{Stab} \mathcal{D}$, the connectedness of $\operatorname{Stab} \mathcal{D}$ follows.

Theorem 7 due to Bridgeland shows that the simply-connectedness of $\operatorname{Stab} \mathcal{D}$ follows from the faithfulness of an affine braid group action on \mathcal{D} . We prove this using homological mirror symmetry for A_n -singularities and ideas from Khovanov and Seidel [33]. Unfortunately, we have to work over a field of characteristic two in order to apply Theorem 27 by Khovanov and Seidel, and we lift this faithfulness result to any characteristic using the deformation theory of complexes by Inaba [29].

In contrast to the case of $\operatorname{Stab} \mathcal{D}$, we cannot use algebro-geometric argument in the proof of the connectedness of $\operatorname{Stab} \mathcal{C}$, since \mathcal{C} does not contain any skyscraper sheaves. Instead, we use a result in [30] and ideas from [33] to reduce the problem of the connectedness of $\operatorname{Stab} \mathcal{C}$ to that of configurations of curves on a disk.

The organization of this paper is as follows: In §2, we collect basic definitions and known results used in this paper. In §3, we recall the McKay correspondence for A_n -singularities in such a way that is valid in any characteristic. In §4, we give the proof of the connectedness of $\operatorname{Stab} \mathcal{D}$. We prove the faithfulness in characteristic two in §5 and lift it to any characteristic in §6. The connectedness of $\operatorname{Stab} \mathcal{C}$ is proved in §7. In the appendix, we prove that every autoequivalence of \mathcal{D} is given by an integral functor.

Acknowledgment: We thank Jun-ichi Matsuzawa and Yukinobu Toda for valuable discussions and suggestions. A. I. is supported by the Grants-in-Aid for Scientific Research (No.18540034). K. U. is supported by the 21st Century COE Program of Osaka University. H. U. is supported by the Grants-in-Aid for Scientific Research (No.17740012).

2 Generalities

We collect basic definitions and known results in this section. All the categories appearing in this paper will be essentially small. For a triangulated category \mathcal{T} , $K(\mathcal{T})$ denotes its Grothendieck group, and for an object $E \in \mathcal{T}$, $[E]$ will denote its class in $K(\mathcal{T})$. For two objects $E, F \in \mathcal{T}$ and $i \in \mathbb{Z}$, $\text{Hom}_{\mathcal{T}}^*(E, F)$, $\text{Hom}_{\mathcal{T}}^{\leq i}(E, F)$, and $\text{Hom}_{\mathcal{T}}^{\geq i}(E, F)$ will denote $\bigoplus_{j \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}^j(E, F)$, $\bigoplus_{j \leq i} \text{Hom}_{\mathcal{T}}^j(E, F)$, and $\bigoplus_{j \geq i} \text{Hom}_{\mathcal{T}}^j(E, F)$, respectively.

2.1 Stability conditions on triangulated categories

The following definition is introduced by Bridgeland [9] based on the work of Douglas et al. [2, 15, 16, 17, 18] on the stability of BPS D-branes:

Definition 2. A stability condition $\sigma = (Z, \mathcal{P})$ on a triangulated category \mathcal{T} consists of

- a group homomorphism $Z : K(\mathcal{T}) \rightarrow \mathbb{C}$, and
- full additive subcategories $\mathcal{P}(\phi)$ for $\phi \in \mathbb{R}$

satisfying the following conditions:

- (i) If $0 \neq E \in \mathcal{P}(\phi)$, then $Z(E) = m(E) \exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,
- (ii) for all $\phi \in \mathbb{R}$, $\mathcal{P}(\phi + 1) = \mathcal{P}(\phi)[1]$,
- (iii) for $A_j \in \mathcal{P}(\phi_j)$ ($j = 1, 2$) with $\phi_1 > \phi_2$, we have $\text{Hom}_{\mathcal{T}}(A_1, A_2) = 0$,
- (iv) for every non-zero object $E \in \mathcal{T}$, there is a finite sequence of real numbers

$$\phi_1 > \phi_2 > \cdots > \phi_n$$

and a collection of triangles

$$\begin{array}{ccccccc} 0 = E_0 & \xrightarrow{\quad} & E_1 & \xrightarrow{\quad} & E_2 & \cdots \rightarrow & E_{n-1} & \xrightarrow{\quad} & E_n = E \\ & \swarrow \text{dashed} & \searrow & & \swarrow \text{dashed} & & \searrow & & \swarrow \text{dashed} \\ & & A_1 & & A_2 & & & & A_n \end{array} \quad (1)$$

with $A_j \in \mathcal{P}(\phi_j)$ for all j .

Z is called the *central charge*, and the collection of triangles in (1) is called the *Harder-Narasimhan filtration*. It follows from the definition that $\mathcal{P}(\phi)$ is an abelian category, and its non-zero object $\mathcal{E} \in \mathcal{P}(\phi)$ is said to be *semistable of phase ϕ* . \mathcal{E} is said to be *stable* if it is a simple object of $\mathcal{P}(\phi)$, i.e., there are no proper subobjects of \mathcal{E} in $\mathcal{P}(\phi)$. By [9, proposition 5.3], to give a stability condition on a triangulated category \mathcal{T} is equivalent to giving a bounded t -structure on \mathcal{T} and a *stability function* (previously called

a centered slope-function) on its heart with the *Harder-Narasimhan property*. For the definitions of a stability function and the Harder-Narasimhan property, see [9, §2].

The set of stability conditions satisfying a certain technical condition called *local-finiteness* [9, definition 5.7] is denoted by $\text{Stab } \mathcal{T}$. This condition ensures that each $\mathcal{P}(\phi)$ is a finite length category so that each semi-stable object has a Jordan-Hölder filtration. By combining it with the Harder-Narasimhan filtration, any non-zero object $E \in \mathcal{T}$ admits a decomposition as in (1) such that $A_j \in \mathcal{P}(\phi_j)$ is stable for all j and $\phi_1 \geq \phi_2 \geq \dots \geq \phi_n$. Bridgeland introduces a natural topology on $\text{Stab } \mathcal{T}$ such that the forgetful map

$$\begin{array}{ccc} \mathcal{Z} : \text{Stab } \mathcal{T} & \rightarrow & \text{Hom}(K(\mathcal{T}), \mathbb{C}) \\ \downarrow \Psi & & \downarrow \Psi \\ (Z, \mathcal{P}) & \mapsto & Z \end{array}$$

satisfies the following:

Theorem 3 ([9, theorem 1.2]). *For each connected component Σ of $\text{Stab } \mathcal{T}$, there is a linear subspace $V(\Sigma) \subset \text{Hom}(K(\mathcal{T}), \mathbb{C})$ with a well-defined linear topology such that the restriction $\mathcal{Z}|_{\Sigma}$ gives a local homeomorphism.*

Hence $\text{Stab } \mathcal{T}$ forms a (possibly infinite-dimensional) complex manifold modeled on the topological vector space $V(\Sigma)$. When $\mathcal{T} = \mathcal{C}$ or \mathcal{D} , $K(\mathcal{T})$ is finite-dimensional, and we prove in Lemma 15 that $V(\Sigma)$ always coincides with $\text{Hom}(K(\mathcal{T}), \mathbb{C})$.

Since the definition of $\text{Stab } \mathcal{T}$ uses only the triangulated structure of \mathcal{T} , the group $\text{Auteq } \mathcal{T}$ of triangle autoequivalences of \mathcal{T} acts naturally on $\text{Stab } \mathcal{T}$ from the left; for $\sigma = (Z, \mathcal{P}) \in \text{Stab } \mathcal{T}$ and $\Phi \in \text{Auteq } \mathcal{T}$,

$$\Phi(\sigma) = (\Phi^* Z, \Phi(\mathcal{P}))$$

where Φ^* is the pull-back by the inverse of the automorphism $\Phi_* : K(\mathcal{T}) \rightarrow K(\mathcal{T})$ induced by Φ . This action commutes with the right action of the universal cover $\widetilde{GL}^+(2, \mathbb{R})$ of the general linear group $GL^+(2, \mathbb{R})$ with positive determinant, which “rotates” the central charge [9, lemma 8.2].

2.2 Minimal resolutions of A_n -singularities

We consider an arbitrary field k . The case $\text{char}(k) = 2$ will be important later. For a positive integer n , let

$$f : X \rightarrow \text{Spec } k[x, y, z]/(xy + z^{n+1})$$

be the minimal resolution of the A_n -singularity. The exceptional set of f will be denoted by

$$Z = f^{-1}(0) = C_1 \cup \dots \cup C_n,$$

where C_i 's are irreducible (-2) -curves such that $C_i \cap C_j = \emptyset$ if $|i-j| > 1$. Let \mathcal{D}_k be the bounded derived category of coherent sheaves on X supported at Z and \mathcal{C}_k be its full triangulated subcategory consisting of objects E satisfying $\mathbb{R}f_*E = 0$. Put $E_0 = \omega_Z$ and $E_i = \mathcal{O}_{C_i}(-1)$ for $i = 1, \dots, n$. Here, ω_Z is the dualizing sheaf of Z . Then we have

$$\mathcal{C}_k = \langle E_1, \dots, E_n \rangle$$

and

$$\mathcal{D}_k = \langle E_0, \dots, E_n \rangle,$$

where $\langle \bullet \rangle$ denotes the smallest full triangulated subcategory of \mathcal{D}_k containing them. We simply write \mathcal{C} and \mathcal{D} instead of $\mathcal{C}_{\mathbb{C}}$ and $\mathcal{D}_{\mathbb{C}}$, respectively.

For $E, F \in \mathcal{D}_k$, define the Euler form by

$$\chi(E, F) = \sum_i (-1)^i \dim \operatorname{Hom}_{\mathcal{D}_k}^i(E, F), \quad (2)$$

which descends to a bilinear form on $K(\mathcal{D}_k)$. By the Riemann-Roch formula, we have

$$\chi(E, F) = -c_1(E) \cdot c_1(F).$$

The Euler form χ endows $K(\mathcal{D}_k)$ with the structure of the affine root lattice of type $A_n^{(1)}$. A non-zero element $\alpha \in K(\mathcal{D})$ is a *root* if $\chi(\alpha, \alpha) \leq 2$ and it is a *real root* if $\chi(\alpha, \alpha) = 2$. An *imaginary root* is a root that is not a real root. Let $\delta \in K(\mathcal{D}_k)$ be the class of the structure sheaf of a closed point with residue field k . Then an imaginary root is a non-zero element of $\mathbb{Z}\delta \subset K(\mathcal{D}_k)$.

Lemma 4. *If $E \in \mathcal{D}$ is stable with respect to some stability condition, then $[E] \in K(\mathcal{D})$ is a root.*

Proof. The stability of E implies $\operatorname{Hom}_{\mathcal{D}}^{\leq -1}(E, E) = 0$ and $\operatorname{Hom}_{\mathcal{D}}(E, E) \cong \mathbb{C}$. The Serre duality shows $\operatorname{Hom}_{\mathcal{D}}^{\geq 3}(E, E) = 0$ and $\operatorname{Hom}_{\mathcal{D}}^2(E, E) \cong \mathbb{C}$. Hence $\chi(E, E) \leq 2$ and $[E]$ is a root. \square

Definition 5. (i) An object $E \in \mathcal{D}_k$ is spherical if

$$\operatorname{Hom}_{\mathcal{D}_k}^i(E, E) \cong \begin{cases} k & \text{if } i = 0, 2, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) An ordered set (E_1, \dots, E_n) of spherical objects in \mathcal{D}_k is an A_n -configuration if

$$\dim \operatorname{Hom}_{\mathcal{D}_k}^*(E_i, E_j) = \begin{cases} 1 & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| \geq 2. \end{cases}$$

The proof of Lemma 4 shows the following:

Lemma 6. *If the class $[E] \in K(\mathcal{D})$ of a stable object $E \in \mathcal{D}$ is a real root, then E is spherical.*

A spherical object $E \in \mathcal{D}_k$ gives rise to an autoequivalence of \mathcal{D}_k through the *twist functor* T_E , defined as the Fourier-Mukai transform with

$$\{E^\vee \boxtimes E \rightarrow \mathcal{O}_\Delta\} \in D^b \text{coh } X \times X$$

as the kernel [41]. Define

$$\text{Br}(\mathcal{D}_k) = \langle T_{E_0}, \dots, T_{E_n} \rangle \subset \text{Auteq } \mathcal{D}_k$$

and

$$\text{Br}(\mathcal{C}_k) = \langle T_{E_1}, \dots, T_{E_n} \rangle \subset \text{Auteq } \mathcal{C}_k.$$

Define the braid group B_n as the group generated by $\sigma_1, \dots, \sigma_n$ subject to relations

$$\begin{aligned} \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & i &= 1, \dots, n-1, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i-j| &> 2. \end{aligned}$$

It has the following topological description: Let

$$\mathfrak{h} = \{(a_1, \dots, a_{n+1}) \in \mathbb{C}^{n+1} \mid a_1 + \dots + a_{n+1} = 0\}$$

be a Cartan subalgebra of the complex simple Lie algebra of type A_n , and $\mathfrak{h}^{\text{reg}}$ be the complement of its root hyperplanes,

$$\mathfrak{h}^{\text{reg}} = \{(a_1, \dots, a_{n+1}) \in \mathfrak{h} \mid a_i \neq a_j \text{ for } i \neq j\}.$$

The Weyl group $W \cong \mathfrak{S}_{n+1}$ acts on \mathfrak{h} by permutations, and $\mathfrak{h}^{\text{reg}}$ is the set of regular orbits of W . Then B_n is isomorphic to the fundamental group of the quotient $\mathfrak{h}^{\text{reg}}/W$ [13, 14]. It follows that B_n has another topological description: Let $\Delta = \{1, \zeta, \zeta^2, \dots, \zeta^n\}$ be the set of $(n+1)$ th roots of unity and $\text{Diff}_0(\mathbb{C})$ be the group of diffeomorphisms of \mathbb{C} which are the identity map outside compact sets. Then there is a map $\text{Diff}_0(\mathbb{C}) \rightarrow \mathfrak{h}^{\text{reg}}/W$ which sends $\phi \in \text{Diff}_0(\mathbb{C})$ to $[\{\phi(1) - c, \phi(\zeta) - c, \dots, \phi(\zeta^n) - c\}]$ with $c = \sum_{i=0}^n \phi(\zeta^i)/(n+1)$. This map is a Serre fibration whose fiber over $[\Delta]$ is the subgroup $\text{Diff}_0(\mathbb{C}; \Delta) \subset \text{Diff}_0(\mathbb{C})$ which fixes Δ as a set. From the long exact sequence of homotopy groups associated to this fibration, we can see that

$$B_n \cong \pi_0(\text{Diff}_0(\mathbb{C}; \Delta)).$$

The assignment $\sigma_i \mapsto T_{E_i}$ for $i = 1, \dots, n$ defines a homomorphism from B_n to $\text{Br}(\mathcal{C})$, which is injective by Khovanov, Seidel, and Thomas [33, 41]. This result is the key to the proof of the simply-connectedness of a distinguished connected component of $\text{Stab } \mathcal{C}$ by Thomas [42].

Now define the affine braid group $B_n^{(1)}$ to be the group generated by $\sigma_0, \dots, \sigma_n$ subject to relations

$$\begin{aligned}\sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & i = 0, \dots, n, \\ \sigma_i \sigma_j &= \sigma_j \sigma_i, & |i - j| > 2.\end{aligned}$$

Here, we put $\sigma_{n+1} = \sigma_0$ by notation. Let $\widehat{\mathfrak{h}}^{\text{reg}}$ be the complement of the affine root hyperplanes in $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}$:

$$\widehat{\mathfrak{h}}^{\text{reg}} = \{(a_1, \dots, a_{n+1}, b) \in \mathfrak{h} \oplus \mathbb{C} \mid a_i - a_j + bd \neq 0 \text{ for } i \neq j \text{ and } d \in \mathbb{Z}\}.$$

The affine Weyl group \widehat{W} acts freely on $\widehat{\mathfrak{h}}^{\text{reg}}$, and the fundamental group of the orbit space $\widehat{\mathfrak{h}}^{\text{reg}}/\widehat{W}$ is given by $B_n^{(1)} \times \mathbb{Z}$ by [19].

The group $B_n^{(1)}$ also admits the following topological interpretation: Let $\text{Diff}_0(\mathbb{C}^\times)$ be the group of diffeomorphisms of \mathbb{C}^\times which are the identity maps outside compact sets and $\text{Diff}_0(\mathbb{C}^\times; \Delta)$ be its subgroup fixing Δ as a set. We can define a homomorphism

$$B_n^{(1)} \rightarrow \pi_0(\text{Diff}_0(\mathbb{C}^\times; \Delta))$$

from $B_n^{(1)}$ to the group of connected components of $\text{Diff}_0(\mathbb{C}^\times; \Delta)$ by sending σ_i to the class of a diffeomorphism of \mathbb{C}^\times which permutes two neighboring points ζ^i and ζ^{i+1} for $i = 0, \dots, n$. This homomorphism is known to be injective (cf. [32]).

The assignment $\sigma_i \mapsto T_{E_i}$ for $i = 0, \dots, n$ defines a homomorphism

$$\rho : B_n^{(1)} \rightarrow \text{Br}(\mathcal{D}_k),$$

which can be extended to a surjective homomorphism

$$\tilde{\rho} : B_n^{(1)} \times \mathbb{Z} \rightarrow \text{Br}(\mathcal{D}_k) \times \mathbb{Z}.$$

Here, since $\text{Br}(\mathcal{D}_k)$ does not contain any power of the shift functor, the right-hand side is considered as a subgroup of $\text{Auteq } \mathcal{D}_k$ so that the second factor \mathbb{Z} corresponds to the group generated by the shift functor [2]. In [11], Bridgeland shows the following theorem for any Kleinian singularities, that is, rational double points over \mathbb{C} .

Theorem 7 ([11, theorem 1.3]). *There is a connected component of $\text{Stab } \mathcal{D}$ which is a covering space of $\widehat{\mathfrak{h}}^{\text{reg}}/\widehat{W}$ such that the group $\text{Br}(\mathcal{D}) \times \mathbb{Z}$ acts as the group of deck transformations.*

Hence we have the canonical group homomorphism

$$\pi_1(\widehat{\mathfrak{h}}^{\text{reg}}/\widehat{W}) = B_n^{(1)} \times \mathbb{Z} \rightarrow \text{Br}(\mathcal{D}) \times \mathbb{Z}.$$

Bridgeland also shows that this homomorphism coincides with $\tilde{\rho}$. Therefore if ρ is injective, we conclude that the connected component in Theorem 7 is simply connected.

We prove the connectedness of $\text{Stab } \mathcal{D}$ in §4 and the injectivity of ρ in §5 and §6. These results, together with the above theorem of Bridgeland, gives the following explicit description of $\text{Stab } \mathcal{D}$:

Theorem 8. *$\text{Stab } \mathcal{D}$ is the universal cover of $\widehat{\mathfrak{h}^{\text{reg}}/W}$.*

As for $\text{Stab } \mathcal{C}$, a result of Thomas [42] shows that there is a distinguished connected component of $\text{Stab } \mathcal{C}$ which is the universal cover of $\mathfrak{h}^{\text{reg}}/W$. We will prove the connectedness of $\text{Stab } \mathcal{C}$ in §7, so that this connected component is the whole of $\text{Stab } \mathcal{C}$.

3 The McKay correspondence

We collect basic facts on the McKay correspondence in this section. We expect that the result in this section is well-known to experts, although we have been unable to locate an appropriate reference. Throughout this section, k will denote a field of any characteristic. We restrict our discussion to the case of A_n -singularities since it is the only case in need in this paper. For a noetherian k -algebra A , the abelian category of finitely generated right A -modules will be denoted by $\text{mod } A$.

3.1 Path algebra and the endomorphism algebra of a reflexive module

Let $\mathcal{A}_n^{(1)}$ be the preprojective algebra for the affine Dynkin quiver of type $A_n^{(1)}$, described explicitly as follows: As a k -vector space, $\mathcal{A}_n^{(1)}$ is generated by the symbols $(i_1|i_2|\dots|i_l)$ for $l \geq 1$, $i_m \in \mathbb{Z}/(n+1)\mathbb{Z}$ and $i_{m+1} = i_m \pm 1$. The multiplication is defined by

$$(i_1|\dots|i_l)(j_1|\dots|j_m) = \begin{cases} (i_1|\dots|i_l|j_2|\dots|j_m) & \text{if } i_l = j_1, \\ 0 & \text{otherwise,} \end{cases}$$

and the relations are generated by

$$(i|i+1|i) = (i|i-1|i)$$

for $i \in \mathbb{Z}/(n+1)\mathbb{Z}$.

Let $\mathcal{O} = k[x, y, z]/(xy + z^{n+1})$ be the affine coordinate ring of the rational double point of type A_n . For an integer $a = (n+1)q + r$ with $0 \leq r \leq n$, consider the fractional ideal $I_a = (y^{q+1}, y^q z^r)\mathcal{O}$ of \mathcal{O} . I_a 's are reflexive \mathcal{O} -modules such that $I_a \cong I_b$ if and only if $a - b$ is divisible by $n+1$. For $i \in \mathbb{Z}/(n+1)\mathbb{Z}$, we lift i to $a \in \mathbb{Z}$ with $0 \leq a \leq n$ and put $E_i = I_a$.

For an integer $b = q(n+1) + r$ with $0 \leq r \leq n$, we fix the isomorphism $I_b \cong E_{(b \bmod n+1)} = I_r$ given by the multiplication by y^{-q} . Consider the reflexive \mathcal{O} -module

$$E = \bigoplus_{i \in \mathbb{Z}/(n+1)\mathbb{Z}} E_i.$$

E is an $\mathcal{A}_n^{(1)} \otimes_k \mathcal{O}$ -module in the following way. The idempotent (i) acts as the projection of E to I_i . The path $(i|i+1)$ corresponds to the homomorphism $I_a \xrightarrow{z} I_{a+1}$ given by the multiplication by z , where $a \in \mathbb{Z}$ is a lift of i . The path $(i|i-1)$ goes to the inclusion $I_a \hookrightarrow I_{a-1}$. Then it is easy to see that we obtain an $\mathcal{A}_n^{(1)} \otimes_k \mathcal{O}$ -action on E . Thus we have a k -algebra homomorphism

$$\eta : \mathcal{A}_n^{(1)} \rightarrow \text{End}_{\mathcal{O}}(E).$$

Proposition 9. *η is an isomorphism.*

Proof. We first note that an element of $\mathcal{A}_n^{(1)}$ is a k -linear combination of the elements $P(i, l, m)$ defined as follows for $i \in \mathbb{Z}/(n+1)\mathbb{Z}$ and $l, m \in \mathbb{Z}$ with $m \geq 0$:

$$P(i, l, m) = \begin{cases} (i|i+1|i)^m (i|i+1|\dots|i+l-1|i+l) & \text{if } l \geq 0, \\ (i|i+1|i)^m (i|i-1|\dots|i+l+1|i+l) & \text{otherwise.} \end{cases}$$

To show that η is an isomorphism, it is sufficient to show that the restricted map

$$\eta_{ij} : (i)\mathcal{A}_n^{(1)}(j) \rightarrow \text{Hom}_{\mathcal{O}}(E_i, E_j)$$

is bijective for $i, j \in \mathbb{Z}/(n+1)\mathbb{Z}$. Lift i, j to $a, a+r \in \mathbb{Z}$ with $0 \leq r \leq n$. Then we have isomorphisms

$$\text{Hom}_{\mathcal{O}}(E_i, E_j) \cong \text{Hom}_{\mathcal{O}}(I_a, I_{a+r}) \cong I_r = (y, z^r)\mathcal{O}.$$

If $l = q(n+1) + r$ with $q \geq 0$, then $P(i, l, m)$ is mapped to $x^q z^{m+r} \in I_r$, and if $l = -q(n+1) + r < 0$ with $q > 0$, then $P(i, l, m)$ is mapped to $y^q z^m \in I_r$. Moreover, the monomials $x^q z^{m+r}$ ($q \geq 0, m \geq 0$) and $y^q z^m$ ($q > 0, m \geq 0$) form a k -linear basis of I_r . Therefore, η_{ij} must be bijective. \square

3.2 A full sheaf as a projective generator

Let $f : X \rightarrow Y = \text{Spec } \mathcal{O}$ be the minimal resolution and ${}^{-1}\text{Per}(X/Y)$ be the abelian category of perverse sheaves introduced by Bridgeland [8]; an object $E \in {}^{-1}\text{Per}(X/Y)$ is a bounded complex of coherent sheaves on X such that its cohomology sheaves satisfy

$$f_*(\mathcal{H}^{-1}(E)) = 0, \quad R^1 f_*(\mathcal{H}^0(E)) = 0, \quad \mathcal{H}^i(E) = 0 \text{ for } i \neq -1, 0,$$

and

$$\text{Hom}_X(\mathcal{H}^0(E), F) = 0$$

for any coherent sheaf F on X satisfying $\mathbb{R}f_*F = 0$.

For a reflexive \mathcal{O} -module F , put

$$\tilde{F} := f^*(F)/\text{torsion},$$

which is a locally free sheaf on X (see [1]). A locally free sheaf of this form is called a *full sheaf*. It is proved in [20] that a locally free sheaf \mathcal{F} on X is a full sheaf if and only if the following two conditions are satisfied:

- (i) \mathcal{F} is generated by its global sections.
- (ii) $R^1f_*(\mathcal{F}^\vee) = 0$.

The following result is due to Van den Bergh:

Proposition 10 ([43, proposition 3.2.7, corollary 3.2.8]). *(i) A full sheaf \mathcal{M} is a projective generator of ${}^{-1}\text{Per}(X/Y)$ if and only if its first Chern class $c_1(\mathcal{M})$ is ample and \mathcal{O}_X is a direct summand of $\mathcal{M}^{\oplus a}$ for some positive integer a .*

(ii) Assume that a full sheaf \mathcal{M} is a projective generator of ${}^{-1}\text{Per}(X/Y)$ and put $A = \text{End}_X(\mathcal{M})$. Then the functor $\mathbb{R}\text{Hom}_X(\mathcal{M}, \bullet)$ gives an equivalence between $D^b \text{coh } X$ and $D^b \text{mod } A$, whose inverse is given by the functor $\bullet \otimes_A^{\mathbb{L}} \mathcal{M}$.

The above proposition yields the following:

Theorem 11. *The bounded derived category $D^b \text{coh } X$ of coherent sheaves on the crepant resolution X of the A_n -singularity is equivalent to the bounded derived category $D^b \text{mod } \mathcal{A}_n^{(1)}$ of finitely generated right $\mathcal{A}_n^{(1)}$ -modules.*

Proof. Put $E = \bigoplus_{i \in \mathbb{Z}/n\mathbb{Z}} E_i$ be as in the previous subsection. Since

$$c_1(\tilde{E}_i) \cdot C_j = \delta_{ij},$$

the corresponding full sheaf \tilde{E} has an ample first Chern class and is hence a projective generator of ${}^{-1}\text{Per}(X/Y)$ satisfying $\text{End}(\tilde{E}) \cong \mathcal{A}_n^{(1)}$. \square

Let $\text{mod}_0 \mathcal{A}_n^{(1)}$ be the abelian category of finitely generated nilpotent right $\mathcal{A}_n^{(1)}$ -modules. Under the above equivalence, \mathcal{D}_k corresponds to the bounded derived category $D^b \text{mod}_0 \mathcal{A}_n^{(1)}$ of $\text{mod}_0 \mathcal{A}_n^{(1)}$.

When k contains a primitive $(n+1)$ th root ζ of unity, Y is isomorphic to the quotient \mathbb{A}^2/G of the affine plane by the natural action of the subgroup G of $SL_2(k)$ generated by the diagonal matrix $\text{diag}(\zeta, \zeta^n)$. Since $\mathcal{A}_n^{(1)}$ is isomorphic to the crossed product $k[x, y] \rtimes k[G]$ of the polynomial ring with the group ring, the category of finitely generated nilpotent $\mathcal{A}_n^{(1)}$ -modules is

equivalent to the category $\mathrm{coh}_0^G \mathbb{A}^2$ of G -equivariant coherent sheaves on \mathbb{A}^2 supported at the origin:

$$\mathrm{mod}_0 \mathcal{A}_n^{(1)} \cong \mathrm{coh}_0^G \mathbb{A}^2.$$

Hence Theorem 11 in this case gives the equivalence

$$D^b \mathrm{coh}_Z X \cong D^b \mathrm{coh}_0^G \mathbb{A}^2$$

of triangulated categories, first proved by Kapranov and Vasserot [31] (see also Bridgeland, King, and Reid [12]).

4 Connectedness of $\mathrm{Stab} \mathcal{D}$

We prove the connectedness of $\mathrm{Stab} \mathcal{D}$ in this section. Our strategy is the following:

(i) Put

$$U := \{ \sigma \in \mathrm{Stab} \mathcal{D} \mid \mathcal{O}_x \text{ is } \sigma\text{-stable for all the closed points } x \in Z \}.$$

Then U is connected. The proof is parallel to the $K3$ case due to Bridgeland [10, §11].

- (ii) The connected component of $\mathrm{Stab} \mathcal{D}$ containing U is preserved by the action of $\mathrm{Br}(\mathcal{D})$. This follows from [11, theorem 1.1] since U belongs to the distinguished connected component studied by Bridgeland.
- (iii) For any connected component Σ of $\mathrm{Stab} \mathcal{D}$, there is a stability condition $\sigma = (Z, \mathcal{P}) \in \Sigma$ such that $Z(\mathcal{O}_x) \in \mathbb{R}_{<0}$ for a closed point $x \in Z$ and $Z(E) \notin \mathbb{R}$ for any spherical object E .
- (iv) For a stability condition σ as above, there is a σ -stable object $\omega \in \mathcal{P}(1)$ satisfying some technical conditions.
- (v) The technical conditions together with a result in [30] imply the existence of a closed point $x \in Z$, an autoequivalence $\Phi \in \mathrm{Br}(\mathcal{D})$, and an integer $d \in \mathbb{Z}$ such that $\omega = \Phi(\mathcal{O}_x[d])$.
- (vi) If \mathcal{O}_x is stable with respect to a stability condition σ for some point $x \in C_i$, then σ induces stability conditions on $D^b \mathrm{coh}_{Z'} X$ and $D^b \mathrm{coh}_{Z''} X$, where $D^b \mathrm{coh}_{Z'} X$ and $D^b \mathrm{coh}_{Z''} X$ are the full triangulated subcategories of \mathcal{D} consisting of objects supported at $Z' = C_1 \cup \cdots \cup C_{i-1}$ and $Z'' = C_{i+1} \cup \cdots \cup C_n$, respectively.
- (vii) This proves the existence of an autoequivalence $\Phi \in \mathrm{Br}(\mathcal{D})$ such that $\Phi\sigma$ belongs to U by induction on n .

Note that the above argument except for (v) works not only for A_n -singularities but also for any rational double points in dimension two.

4.1 The connected component containing U

Here we collect basic results on the connected component of $\text{Stab } \mathcal{D}$ containing a stability condition where the structure sheaves of all the closed points are stable. The following lemma is taken from [10, lemma 10.1]:

Lemma 12. *For a stability condition $\sigma = (Z, \mathcal{P})$ and a closed point $x \in Z$, assume that \mathcal{O}_x is σ -stable of phase 1.*

- (i) *If $E \in \mathcal{P}((0, 1])$, then E is a sheaf in a neighborhood of x .*
- (ii) *If $E \in \mathcal{P}(1)$ and E is stable, then $E = \mathcal{O}_x$ or $x \notin \text{Supp } E$.*

The following two lemmas are also essentially due to Bridgeland, whose proofs we include for completeness. The proof of Lemma 13 is similar to, but easier than, the corresponding statement for K3 surfaces in [10, §11]. As for Lemma 14, we use the result in [11] rather than imitate the proof in [10, §12].

Lemma 13. *The subset $U \subset \text{Stab } \mathcal{D}$ defined by*

$$U := \{ \sigma \in \text{Stab } \mathcal{D} \mid \mathcal{O}_x \text{ is } \sigma\text{-stable for all the closed points } x \in Z \}$$

is connected.

Proof. Let us fix a point $y \in Z$. Consider a stability condition $(Z, \phi) \in U$ that satisfies $\mathcal{O}_y \in \mathcal{P}(1)$ and $Z(\mathcal{O}_y) = -1$. Then, for any point $x \in Z$, $Z(\mathcal{O}_x) = Z(\mathcal{O}_y) = -1$ implies $\phi(\mathcal{O}_x) \in 2\mathbb{Z} + 1$. Suppose y is contained in an irreducible component C_i of Z . Then the support of at least one Harder-Narasimhan factor of \mathcal{O}_{C_i} contains C_i . Hence there is a stable object $E \in \mathcal{P}((0, 1])$ whose support contains C_i . E is a sheaf in a neighborhood of y by Lemma 12 and therefore for a point x in a neighborhood of y in C_i , $\phi(\mathcal{O}_x)$ must be 1. Since Z is connected, $\phi(\mathcal{O}_x) = 1$ holds for every point $x \in Z$ and (Z, ϕ) belongs to the following subset V of U :

$$V := \{ (Z, \mathcal{P}) \in U \mid \mathcal{O}_x \in \mathcal{P}(1) \text{ and } Z(\mathcal{O}_x) = -1 \text{ for any point } x \in Z \}.$$

Therefore we can take any stability conditions in U into V by the action of the connected group $\widetilde{GL}^+(2, \mathbb{R})$. Hence it suffices to show that V is connected. Take a stability condition $\sigma = (Z, \mathcal{P}) \in V$. Lemma 12 implies that the heart $\mathcal{P}((0, 1])$ coincides with the abelian category $\text{coh}_Z X$ and every one-dimensional sheaf in $\text{coh}_Z X$ has the phase ϕ with $0 < \phi < 1$. It follows that Z belongs to the subset \mathcal{L} of $\text{Hom}(K(\mathcal{D}), \mathbb{C})$ defined by

$$\mathcal{L} := \{ Z_{(\beta, \omega)} \in \text{Hom}(K(\mathcal{D}), \mathbb{C}) \mid \beta \in N^1(X/Y), \omega \in \mathcal{A}(X/Y) \}.$$

Here $\mathcal{A}(X/Y)$ is the ample cone in $N^1(X/Y)$ and

$$Z_{(\beta, \omega)}(E) = -ch_2(E) + (\beta + \sqrt{-1}\omega) \cdot c_1(E)$$

for $E \in K(\mathcal{D})$. Now consider the restriction $\mathcal{Z}|_V : V \rightarrow \mathcal{L}$ of the map \mathcal{Z} in Theorem 3, which is clearly injective. We will show in the proof of Lemma 15 that $V(\Sigma) = \text{Hom}(K(\mathcal{D}), \mathbb{C})$ for any connected component Σ of $\text{Stab } \mathcal{D}$. Therefore $\mathcal{Z}|_V$ is also a local homeomorphism. To show that it is surjective, take any $Z_{(\beta, \omega)}$ in \mathcal{L} . Since ω is ample, $Z_{(\beta, \omega)}$ is a stability function on $\text{coh}_Z X$. To show that it has the Harder-Narasimhan property, we check the conditions (a) and (b) in [11, proposition 2.4]. Condition (a) follows from the fact that for any infinite sequence

$$\cdots \subset E_{j+1} \subset E_j \subset \cdots \subset E_2 \subset E_1$$

of proper subobjects, there is a natural number N such that for any $l > m > N$, the difference $[E_m] - [E_l]$ is a positive multiple of δ defined in page 5. Since $\text{coh}_Z X$ is noetherian, there is no infinite sequence of proper quotients and the condition (b) is automatic. Local-finiteness also follows from a similar reasoning. Hence $\mathcal{Z}|_V$ gives a homeomorphism from V to \mathcal{L} , which is connected. \square

Lemma 14. *The connected component of $\text{Stab } \mathcal{D}$ containing U is preserved by the action of $\text{Br}(\mathcal{D})$.*

Proof. Take $\omega \in \mathcal{A}(X/Y)$ satisfying $\omega \cdot C_j = 1$ for all j . Then there is a unique stability condition $\sigma = (Z, \mathcal{P}) \in V$ such that $Z = \exp(\sqrt{-1}\omega)$. This stability function satisfies $Z(\mathcal{O}_x) = -1$, $Z(\mathcal{O}_{C_j}(-1)) = \sqrt{-1}$ for all j and $Z(\omega_Z) = -1 + n\sqrt{-1}$. Moreover, we can easily check that the sheaves $\mathcal{O}_{C_j}(-1)$ for $j = 1, \dots, n$ and ω_Z are σ -stable. Thus, if we take $\alpha \in (0, 1/2)$ with $\tan(\pi\alpha) > n$, then the abelian category $\mathcal{P}((\alpha, \alpha + 1])$ contains $\omega_Z[1], \mathcal{O}_{C_1}(-1), \dots, \mathcal{O}_{C_n}(-1)$, which corresponds to the simple $\mathcal{A}_n^{(1)}$ -modules by the McKay correspondence. Since both are the hearts of bounded t-structures, they must coincide:

$$\mathcal{P}((\alpha, \alpha + 1]) = \text{mod } \mathcal{A}_n^{(1)}.$$

This shows that σ lies in the distinguished connected component appearing in the work of Bridgeland [11]. Hence it is preserved by the action of $\text{Br}(\mathcal{D})$ [11, theorem 1.4]. \square

4.2 Perturbation lemma

Here we prove the following lemma. Recall that $\delta \in K(\mathcal{D})$ is the class of the structure sheaf of a closed point.

Lemma 15. *For any connected component Σ of $\text{Stab } \mathcal{D}$, there is a stability condition $\sigma = (Z, \mathcal{P}) \in \Sigma$ such that $Z(\delta) \in \mathbb{R}_{<0}$ and $Z(E) \notin \mathbb{R}$ for any spherical objects E .*

Proof. Recall that in the proof of Theorem 3, Bridgeland describes $V(\Sigma)$ explicitly as

$$V(\Sigma) = \{U \in \text{Hom}(K(\mathcal{D}), \mathbb{C}) \mid \|U\|_\sigma < \infty\},$$

where

$$\|U\|_\sigma = \sup \left\{ \left| \frac{U(E)}{Z(E)} \right| \mid E \text{ is stable in } \sigma \right\}$$

for a stability condition $\sigma = (Z, \mathcal{P}) \in \Sigma$. He also shows that $V(\Sigma)$ does not depend on the choice of σ .

Now we show that for any connected component Σ of $\text{Stab } \mathcal{D}$, $V(\Sigma)$ coincides with $\text{Hom}(K(\mathcal{D}), \mathbb{C})$. Let $\sigma = (Z, \mathcal{P})$ be a stability condition in Σ . We want to show that $\|U\|_\sigma < \infty$ for every $U \in \text{Hom}(K(\mathcal{D}), \mathbb{C})$.

We first consider the case where $Z(\delta) = 0$. In this case, every stable object must be spherical and its class in the finite root lattice $K(\mathcal{D})/\mathbb{Z}\delta$ is a root. Assume that E_1 and E_2 are stable objects from the heart $\mathcal{P}((0, 1])$ such that their classes in $K(\mathcal{D})/\mathbb{Z}\delta$ coincide. Then we have $\chi(E_1, E_2) = 2$ and $Z(E_1) = Z(E_2)$. It follows that E_1 and E_2 have the same phase and there is a non-zero morphism between them. If this happens for stable objects E_1 and E_2 , then we must have $E_1 \cong E_2$. Thus for every root in $K(\mathcal{D})/\mathbb{Z}\delta$, there is at most one corresponding stable object in $\mathcal{P}((0, 1])$. This shows that there are only finitely many isomorphism classes of stable objects in $\mathcal{P}((0, 1])$. Therefore, we have $\|U\|_\sigma < \infty$ for every U .

Consider the remaining case: $Z(\delta) \neq 0$. For a stable object, its class in the affine root lattice $K(\mathcal{D})$ is a root. Hence there are finitely many stable objects E_1, \dots, E_m such that for every stable E , there exists E_j with $[E] - [E_j] \in \mathbb{Z}\delta \subseteq K(\mathcal{D})$. Since

$$\lim_{k \rightarrow \pm\infty} \frac{U([E_j] + k\delta)}{Z([E_j] + k\delta)} = \frac{U(\delta)}{Z(\delta)}$$

converges for any $U \in \text{Hom}(K(\mathcal{D}), \mathbb{C})$, we obtain $\|U\|_\sigma < \infty$.

Therefore, we can find a stability condition $\sigma' = (Z', \mathcal{P}') \in \Sigma$ such that $Z'(\delta) \neq 0$. Then we can bring σ' to $\sigma'' = (Z'', \mathcal{P}'') \in \Sigma$ such that $Z''(\delta) \in \mathbb{R}_{<0}$ by the action of $\widetilde{GL}^+(2, \mathbb{R})$. Since the set of classes of spherical objects in $K(\mathcal{D})/\mathbb{Z}\delta$ is finite, we can perturb σ'' further to $\sigma = (Z, \mathcal{P})$ satisfying $Z(E) \notin \mathbb{R}$ for any spherical objects E , while keeping the condition $Z(\delta) \in \mathbb{R}_{<0}$ intact. \square

If σ satisfies the conditions in Lemma 15, then the imaginary part $\Im Z$ defines a Weyl chamber of the finite root lattice $K(\mathcal{D})/\mathbb{Z}\delta$; the simple roots are linearly independent elements $\alpha_1, \dots, \alpha_n \in K(\mathcal{D})/\mathbb{Z}\delta$ with $\Im Z(\alpha_i) > 0$ such that any root $\alpha \in K(\mathcal{D})/\mathbb{Z}\delta$ with $\Im Z(\alpha) > 0$ is a linear combination of $\alpha_1, \dots, \alpha_n$ whose coefficients are non-negative.

4.3 Main technical lemma

Here we prove Lemma 16, which lies at the heart of our proof of the connectedness of $\text{Stab } \mathcal{D}$.

Lemma 16. *Assume that a stability condition $\sigma = (Z, \mathcal{P})$ satisfies the conditions in Lemma 15. Then there are σ -stable spherical objects $E, E' \in \mathcal{P}((0, 1))$ with $\text{Hom}_{\mathcal{D}}^1(E, E') = 0$ and a σ -stable object $\omega \in \mathcal{P}(1)$ which fit into the following short exact sequence in $\mathcal{P}((0, 1])$;*

$$0 \rightarrow E' \rightarrow E \rightarrow \omega \rightarrow 0. \quad (3)$$

Proof. The proof goes in five steps:

Step 1. *There is a σ -stable object $\omega \in \mathcal{P}(1)$ such that $[\omega] \in \mathbb{Z}\delta$.*

We will show that there is a non-zero object in $\mathcal{P}(1)$, and the desired stable object in $\mathcal{P}(1)$ will be obtained as its Jordan-Hölder component.

First note that there are objects E_1, E_2 of the heart $\mathcal{A} := \mathcal{P}((0, 1])$ such that $[E_1] - [E_2] = \delta$ since $K(\mathcal{A}) \cong K(\mathcal{D})$. If E_1 is in $\mathcal{P}(1)$, then we are done. If $E_1 \notin \mathcal{P}(1)$, then $[E_1] \notin \mathbb{Z}\delta$ and hence we have

$$\chi(E_1, E_2) = \chi(E_1, E_1 - \delta) = \chi(E_1, E_1) \geq 2.$$

It follows that at least one of $\text{Hom}_{\mathcal{D}}(E_1, E_2)$ and $\text{Hom}_{\mathcal{D}}(E_2, E_1)$ is non-zero. First assume that there is a non-zero morphism $f : E_1 \rightarrow E_2$ and let F be the image of f in the abelian category $\mathcal{P}((0, 1])$. If $F \in \mathcal{P}(1)$, then we are done. If not, put $E'_1 := \text{Ker } f$ and $E'_2 := \text{Coker } f$ in $\mathcal{P}((0, 1])$. Then $[E'_1] - [E'_2] = [E_1] - [E_2] \in \mathbb{Z}\delta$ and we can repeat the argument. Since $[E_1] - [E'_1] = [F]$ and the images of $[E_1], [E'_1]$, and $[F]$ in the finite root lattice $K(\mathcal{D})/\mathbb{Z}\delta$ are sums of positive roots, this process terminates after finitely many steps. The case $\text{Hom}_{\mathcal{D}}(E_2, E_1) \neq 0$ can be treated similarly. Hence there is a stable object in $\mathcal{P}(1)$. Since we assume σ satisfies the conditions in Lemma 15, the class of an object in $\mathcal{P}(1)$ lies in $\mathbb{Z}\delta$ and Step 1 is proved.

We impose an additional condition on ω , which will be crucial in Step 5. Put

$$a = \min\{l \in \mathbb{Z} \mid \text{there is a non-zero object } \omega \in \mathcal{P}(1) \text{ such that } [\omega] = l\delta\}$$

and let $\omega \in \mathcal{P}(1)$ be an object such that $[\omega] = a\delta$. The minimality of a implies that ω is a simple object of $\mathcal{P}(1)$, which means ω is stable.

Step 2. *There is an object $H \in \mathcal{P}((0, 1))$ such that $\text{Hom}_{\mathcal{D}}(H, \omega) \neq 0$.*

Assume that $\text{Hom}_{\mathcal{D}}(H, \omega) = 0$ for any $H \in \mathcal{P}((0, 1))$. Since $\omega \in \mathcal{P}(1)$ and $H \in \mathcal{P}((0, 1))$, we also have

$$\text{Hom}_{\mathcal{D}}^{\leq 0}(\omega, H) = \text{Hom}_{\mathcal{D}}^{\leq -1}(H, \omega) = 0.$$

Moreover, it follows from $[\omega] \in \mathbb{Z}\delta$ that $\chi(\omega, H) = 0$. Then the Serre duality shows that

$$\mathrm{Hom}_{\mathcal{D}}^*(H, \omega) = \mathrm{Hom}_{\mathcal{D}}^*(\omega, H) = 0.$$

For a stable object $\omega' \in \mathcal{P}(1)$ which is not isomorphic to ω , we have

$$\mathrm{Hom}_{\mathcal{D}}^{\leq 0}(\omega', \omega) = \mathrm{Hom}_{\mathcal{D}}^{\leq 0}(\omega, \omega') = 0.$$

This, together with $\chi(\omega, \omega') = 0$ and the Serre duality, again implies

$$\mathrm{Hom}_{\mathcal{D}}^*(\omega', \omega) = \mathrm{Hom}_{\mathcal{D}}^*(\omega, \omega') = 0.$$

Hence \mathcal{D} admits an orthogonal decomposition

$$\mathcal{D} = \langle \Omega, \Omega^\perp \rangle$$

into $\Omega = \langle \omega \rangle$ and $\Omega^\perp = {}^\perp\Omega$. This is impossible since Z is connected (cf. [7, example 3.2]).

Step 3. *There is an object $E \in \mathcal{P}((0, 1))$ such that $\mathrm{Hom}_{\mathcal{D}}(E, \omega) \neq 0$ and for any non-zero subobject $F \subset E$ in the abelian category $\mathcal{P}((0, 1])$, the quotient object E/F lies in $\mathcal{P}(1)$.*

If H in Step 2 satisfies the second condition in Step 3, put $E = H$. If not, then there is a non-zero subobject $F \subset H$ such that H/F does not lie in $\mathcal{P}(1)$. If H/F is in $\mathcal{P}((0, 1))$, then put $H_1 = F$ and $H_2 = H/F$. If H/F is not in $\mathcal{P}((0, 1))$, then let $H_1 \subset H$ be the pull-back (to H) of the first Harder-Narasimhan factor of H/F and put $H_2 = H/H_1$. Both H_1 and H_2 are in $\mathcal{P}((0, 1))$ and either $\mathrm{Hom}_{\mathcal{D}}(H_1, \omega) \neq 0$ or $\mathrm{Hom}_{\mathcal{D}}(H_2, \omega) \neq 0$ holds. If $\mathrm{Hom}_{\mathcal{D}}(H_i, \omega) \neq 0$, we can replace H by H_i . We have $[H] = [H_1] + [H_2]$ in $K(\mathcal{D})$ and the images of these objects in $K(\mathcal{D})/\mathbb{Z}\delta$ are non-zero sums of positive roots. Thus we cannot repeat the process infinitely many times and Step 3 is proved.

Step 4. *Let $f : E \rightarrow \omega$ be a non-zero morphism and put $E' := \mathrm{Ker} f$ in $\mathcal{P}((0, 1])$. Then E and E' are stable and spherical.*

Since E and E' are not in $\mathcal{P}(1)$, Lemma 6 implies that it suffices to show that they are stable. Let $0 \neq F \subsetneq E$ be a proper subobject. Then by Step 3, E/F is in $\mathcal{P}(1)$. From the equality

$$Z(F) = Z(E) - Z(E/F)$$

and the fact that

$$\Im Z(F) > 0, \quad \Im Z(E) > 0, \quad \text{and} \quad Z(E/F) \in \mathbb{R}_{<0},$$

we obtain $\phi(F) < \phi(E)$. The proof of the stability of E' is the same.

Step 5. $\mathrm{Hom}_{\mathcal{D}}^1(E, E') = 0$.

Assume $\text{Hom}_{\mathcal{D}}^1(E, E') \neq 0$. Then there is a non-trivial extension

$$0 \rightarrow E' \rightarrow L \rightarrow E \rightarrow 0 \quad (4)$$

in $\mathcal{P}((0, 1])$. We will show that L is stable, which contradicts $\chi(L, L) = 8$ and Lemma 4. Let $M \subsetneq L$ be a non-zero proper subobject and put $F' := E' \cap M \subset E'$ and $F := M/F' \subset E$;

$$\begin{array}{ccccccccc} 0 & \rightarrow & F' & \rightarrow & M & \rightarrow & F & \rightarrow & 0 \\ & & \cap & & \cap & & \cap & & \\ 0 & \rightarrow & E' & \rightarrow & L & \rightarrow & E & \rightarrow & 0. \end{array}$$

Then there are three cases to be considered:

- *The case $F \neq 0$ and $F' \neq 0$:*

By Step 3, we have $E'/F', E/F \in \mathcal{P}(1)$. From the equality

$$Z(M) = Z(L) - (Z(E'/F') + Z(E/F))$$

and the fact that $Z(E'/F'), Z(E/F) \in \mathbb{R}_{<0}$, we obtain $\phi(M) < \phi(L)$.

- *The case $F = 0$ and $F' \neq 0$:*

Since E' is stable and $Z(L) = 2Z(E') + Z(\omega)$, we have $\phi(M) = \phi(F') \leq \phi(E') < \phi(L)$.

- *The case $F \neq 0$ and $F' = 0$:*

If $F = E$, then (4) splits and contradicts our assumption. Hence we have $F \neq E$ and $[E/F] = b\delta$ with $b > 0$ by Step 3. Then our choice of ω implies $a \leq b$. From the equality

$$Z(M) = Z(F) = Z(E) - bZ(\delta) = Z(E') + (a - b)Z(\delta),$$

we conclude $\phi(M) \leq \phi(E') < \phi(L)$.

□

4.4 Stabilizing a skyscraper sheaf

The proof of [30, proposition 1.7] actually shows the following stronger result.

Lemma 17. *Let E and E' be spherical objects in \mathcal{D} satisfying*

$$\text{Hom}_{\mathcal{D}}^i(E', E) \cong \begin{cases} \mathbb{C}^{\oplus 2} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then there is an autoequivalence $\Phi \in \text{Br}(\mathcal{D}) \times \mathbb{Z}$ such that $\Phi(E') \cong \mathcal{O}_{C_l}(a-1)$ and $\Phi(E) \cong \mathcal{O}_{C_l}(a)$ for some $l \in \{1, \dots, n\}$ and $a \in \mathbb{Z}$.

Let E, E', ω be the objects obtained in Lemma 16. Then we obtain an autoequivalence $\Phi \in \text{Br}(\mathcal{D}) \times \mathbb{Z}$ by applying Lemma 17 to E and E' . Since ω fits into the short exact sequence (3), we have

$$\Phi(\omega) \cong \{\mathcal{O}_{C_l}(a-1) \rightarrow \mathcal{O}_{C_l}(a)\} \cong \mathcal{O}_y$$

for some point $y \in C_l$. Replacing ω by its shift if necessary, we obtain the following:

Proposition 18. *Let $\sigma = (Z, \mathcal{P})$ be a stability condition satisfying the conditions in Lemma 15. Then there is a σ -stable object $\omega \in \mathcal{D}$ such that $\Phi(\omega) \cong \mathcal{O}_y$ for a closed point $y \in Z$ and an autoequivalence $\Phi \in \text{Br}(\mathcal{D})$.*

4.5 Induction lemma

Here we prove Corollary 21, which states that a stability condition σ which stabilizes the skyscraper sheaf \mathcal{O}_x for some closed point $x \in C_i$ induces a stability condition on the subcategory of \mathcal{D} consisting of objects supported at $Z' = \bigcup_{i \neq j} C_j$. The following lemma is a consequence of Lemma 12:

Lemma 19. *Assume that \mathcal{O}_x is σ -stable for some stability condition σ . For an object $E \in \mathcal{D}$, let*

$$0 = E_0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow \cdots \longrightarrow E_{m-1} \longrightarrow E_m = E$$

\swarrow
 \searrow
 A_1

\swarrow
 \searrow
 A_2

\swarrow
 \searrow
 A_m

be a collection of triangles such that for some real numbers

$$\phi_1 \geq \phi_2 \geq \cdots \geq \phi_m,$$

A_j is in $\mathcal{P}(\phi_j)$ and stable for $j = 1, \dots, m$, obtained by combining the Harder-Narasimhan filtration with a Jordan-Hölder filtration. If $\text{Supp } E$ does not contain x , then $\bigcup_{j=1}^m \text{Supp } A_j$ does not contain x either.

Proof. We may assume that the phase of \mathcal{O}_x is 1 and E is not σ -stable. Suppose for contradiction that $x \in \text{Supp } A_j$ for some j . By replacing E with E_j for a suitable j , we may assume that $x \in \text{Supp } A_m$. Since the morphism $E \rightarrow A_m$ is not zero, we have $\text{Supp } A_m \supsetneq \{x\}$. Then Lemma 12(ii) ensures that $\phi(A_m)$ is not an integer and hence $d < \phi(A_m) < d+1$ for some $d \in \mathbb{Z}$. Since $A_m[-d]$ is a sheaf near x by Lemma 12(i) and $\text{Supp } A_m \ni x$ by our assumption, we obtain $\text{Hom}_{\mathcal{D}}(A_m, \mathcal{O}_x[d]) \neq 0$, which implies $\text{Hom}_{\mathcal{D}}(E_{m-1}, \mathcal{O}_x[d-1]) \neq 0$. This shows that $\text{Hom}_{\mathcal{D}}(A_l, \mathcal{O}_x[d-1]) \neq 0$ for some l with $l < m$. Hence we have $\phi(A_l) \leq \phi(\mathcal{O}_x[d-1]) = d$, which contradicts the fact that $\phi(A_l) \geq \phi(A_m) > d$. \square

The above lemma yields the following two corollaries:

Corollary 20. *Suppose that \mathcal{O}_x is σ -stable for some $x \in C_i$. If a point $y \in C_i$ is not contained in C_j for any $j \neq i$, then \mathcal{O}_y is also σ -stable.*

Proof. We may assume that $x \neq y$. Consider the above filtration for $E = \mathcal{O}_y$. The support of each stable factor A_j is connected by its stability and does not contain x by the above lemma. It follows that $\text{Supp } A_j$ is just the single point y or it does not contain y . On the other hand, since \mathcal{O}_y is indecomposable, the union $\bigcup_j \text{Supp } A_j$ is connected. Therefore, we have $\bigcup_j \text{Supp } A_j = \{y\}$. This shows that A_j must be of the form $\mathcal{O}_y[d]$ for some $d \in \mathbb{Z}$ since we have $\text{Hom}_{\mathcal{D}}(A_j, A_j) \cong \mathbb{C}$ and $\text{Hom}_{\mathcal{D}}^{\leq -1}(A_j, A_j) = 0$ by the stability of A_j . Hence $\mathcal{O}_y \cong A_j[-d]$ is stable. \square

Corollary 21. *Assume that \mathcal{O}_x is σ -stable for some $x \in C_i$, and put $Z' = \bigcup_{j \neq i} C_j$. Then σ -semistable factors of an object supported at Z' are again supported at Z' .*

Proof. Corollary 20 implies that for any $y \in Z \setminus Z'$, \mathcal{O}_y is σ -stable. Then the statement follows from Lemma 19. \square

4.6 The proof of the connectedness of $\text{Stab } \mathcal{D}$

Here we finish the proof of the connectedness of $\text{Stab } \mathcal{D}$.

Lemma 22. *For any connected component Σ of $\text{Stab } \mathcal{D}$, there is a stability condition $\sigma \in \Sigma$ and an autoequivalence $\Phi \in \text{Br}(\mathcal{D})$ such that $\Phi\sigma$ belongs to U .*

Proof. We use the induction on n . The perturbation lemma guarantees the existence of a stability condition $\sigma = (Z, \mathcal{P}) \in \Sigma$ satisfying $Z(\delta) \in \mathbb{R}_{<0}$ and $Z(E) \notin \mathbb{R}$ for any spherical object E , and we wish to find an autoequivalence $\Phi \in \text{Br}(\mathcal{D})$ such that $\Phi\sigma$ belongs to U .

Proposition 18 says that the structure sheaf \mathcal{O}_y of some point $y \in Z$ is stable with respect to $\Phi\sigma$ for some $\Phi \in \text{Br}(\mathcal{D})$. In the case $n = 1$, at this stage, Corollary 20 ensures that $\Phi\sigma$ belongs to U . In the case $n > 1$, assume $y \in C_l$ and put $Z_1 = C_1 \cup \dots \cup C_{l-1}$ and $Z_2 = C_{l+1} \cup \dots \cup C_n$. Moreover we define

$$\text{Br}(\mathcal{D})_1 := \left\langle T_{\mathcal{O}_{C_i}(a)} \mid 1 \leq i \leq l-1, a \in \mathbb{Z} \right\rangle$$

and

$$\text{Br}(\mathcal{D})_2 := \left\langle T_{\mathcal{O}_{C_i}(a)} \mid l+1 \leq i \leq n, a \in \mathbb{Z} \right\rangle.$$

Corollary 21 implies that $\Phi\sigma$ naturally induces stability conditions on $D^b \text{coh}_{Z_1} X$ and on $D^b \text{coh}_{Z_2} X$, respectively. Then by the induction hypothesis, we can find autoequivalences $\Phi_1 \in \text{Br}(\mathcal{D})_1$ and $\Phi_2 \in \text{Br}(\mathcal{D})_2$ such that $\Phi_1 \Phi_2 \Phi\sigma \in U$. Note that we can regard Φ_k as an element of $\text{Auteq } \mathcal{D}$ via the natural inclusion $\text{Br}(\mathcal{D})_j \subset \text{Br}(\mathcal{D})$ for $j = 1, 2$. \square

Lemmas 13, 14, and 22 imply the connectedness of $\text{Stab } \mathcal{D}$.

5 Faithfulness in characteristic two

In this section, we prove the faithfulness of the action of the affine braid group $B_n^{(1)}$ on \mathcal{D}_k in characteristic two. The assumption on the characteristic is needed in Theorem 27 due to Khovanov and Seidel [33]. Throughout this section, k will be a field of characteristic two. We adopt the following approach, which generalizes to the affine case the method of Khovanov, Seidel, and Thomas [33, 41] who proved the faithfulness of the braid group actions:

- (i) Consider the affine manifold

$$W = \{(x, y, z) \in \mathbb{C}^2 \times \mathbb{C}^\times \mid xyz = z^{n+1} - 1\}$$

with a natural exact symplectic structure. It is a conic fibration over the z -plane whose discriminant set is the set of $(n+1)$ th roots of unity. Any curve on the z -plane without self-intersections starting and ending at the discriminant set gives an exact Lagrangian two-sphere of W .

- (ii) There are $n+1$ Lagrangian two-spheres $\{L_i\}_{i=0}^n$ of W constructed from straight line segments $\{c_i\}_{i=0}^n$ on the z -plane connecting the neighboring discriminant points. Let $\mathfrak{Fuk} W$ be the Fukaya category whose set of objects is $\{L_i\}_{i=0}^n$ and whose spaces of morphisms are Lagrangian intersection Floer complexes. Although Fukaya categories are not honest categories but only A_∞ -categories in general, the above $\mathfrak{Fuk} W$ turns out to be a differential graded category with the trivial differential, since there are no non-constant pseudoholomorphic maps from a disk to W with Lagrangian boundary conditions given by $\{L_i\}_{i=0}^n$.
- (iii) Let $D^b \mathfrak{Fuk} W$ be the derived category of $\mathfrak{Fuk} W$ defined using twisted complexes. It is equivalent to the derived category $D^b \text{mod}_0 \mathcal{A}_n^{(1)}$ of finite-dimensional nilpotent representations of the path algebra $\mathcal{A}_n^{(1)}$ with relations appearing in §3. This is the *homological mirror symmetry* of Kontsevich [34] for A_n -singularities.
- (iv) There are two ways to define an action of the affine braid group on $D^b \mathfrak{Fuk} W$; one is algebraic and given by the twist functor, and the other is geometric and given by the symplectic Dehn twist. They coincide by Seidel [39, proposition 9.1], which allows us to use symplectic geometry of W to prove the faithfulness of the affine braid group action.
- (v) The affine braid group also acts on curves on the z -plane. The actions of the affine braid group on curves on the z -plane and Lagrangian submanifolds of W commute with the construction of Lagrangian submanifolds from curves.

- (vi) By Khovanov and Seidel [33, theorem 1.3], the dimension of the Floer cohomology group between two Lagrangian two-spheres coming from two curves on the z -plane is given by twice their geometric intersection number. This step requires us to work in characteristic two.
- (vii) Suppose given an element b of the affine braid group such that the geometric intersection number between $b'(c_i)$ and $b(c_j)$ is equal to that between $b'(c_i)$ and c_j for any $i, j = 0, \dots, n$ and any b' in the affine braid group. Then b is the identity.
- (viii) (vi) and (vii) suffice to prove the faithfulness of the action of the affine braid group on $D^b \mathfrak{Fuk} W$, and hence on $D^b \bmod_0 \mathcal{A}_n^{(1)} \cong \mathcal{D}_k$.

5.1 A_∞ -categories and twisted complexes

Here we recall the rudiments of A_∞ -categories. For a \mathbb{Z} -graded k -vector space $N = \bigoplus_{j \in \mathbb{Z}} N^j$ and an integer i , $N[i]$ denotes its i -shift to the left; $(N[i])^j = N^{i+j}$.

Definition 23. An A_∞ -category \mathcal{A} consists of

- the set $\mathfrak{Ob}(\mathcal{A})$ of objects,
- for $c_1, c_2 \in \mathfrak{Ob}(\mathcal{A})$, a \mathbb{Z} -graded k -vector space $\mathrm{hom}_{\mathcal{A}}(c_1, c_2)$ called the space of morphisms, and
- operations

$$\mathfrak{m}_l : \mathrm{hom}_{\mathcal{A}}(c_{l-1}, c_l)[1] \otimes \cdots \otimes \mathrm{hom}_{\mathcal{A}}(c_0, c_1)[1] \longrightarrow \mathrm{hom}_{\mathcal{A}}(c_0, c_l)[1]$$

of degree $+1$ for $l = 1, 2, \dots$, satisfying the A_∞ -relations

$$\sum_{i=0}^{l-1} \sum_{j=i+1}^l \mathfrak{m}_{l+i-j+1}(a_l \otimes \cdots \otimes a_{j+1} \otimes \mathfrak{m}_{j-i}(a_j \otimes \cdots \otimes a_{i+1})) \otimes a_i \otimes \cdots \otimes a_1 = 0, \quad (5)$$

for any positive integer l , any sequence c_0, \dots, c_l of objects of \mathcal{A} , and any sequence of morphisms $a_i \in \mathrm{hom}_{\mathcal{A}}(c_{i-1}, c_i)$ for $i = 1, \dots, l$.

Since the A_∞ -relation (5) for $l = 1$ ensures that \mathfrak{m}_1 squares to zero, we can define the *cohomology category* $H^0(\mathcal{A})$ of an A_∞ -category \mathcal{A} by $\mathfrak{Ob}(H^0(\mathcal{A})) = \mathfrak{Ob}(\mathcal{A})$ and $\mathrm{hom}_{H^0(\mathcal{A})}(c_0, c_1) = H^0(\mathrm{hom}_{\mathcal{A}}(c_0, c_1), \mathfrak{m}_1)$.

To define the derived category of an A_∞ -category, we need the concept of *twisted complexes*. It is originally due to Bondal and Kapranov [4] in the case of differential graded categories (i.e., when $\mathfrak{m}_k = 0$ for $k \geq 3$), and generalized to A_∞ -categories by Kontsevich [34].

Definition 24. Let \mathcal{A} be an A_∞ -category.

- (i) The *additive enlargement* $\Sigma\mathcal{A}$ is obtained from \mathcal{A} by formally adding direct sums and shifts; an object c of $\Sigma\mathcal{A}$ is a formal sum

$$c = \bigoplus_{i \in I} c_i[l_i]$$

where I is a finite index set, $c_i \in \mathfrak{Ob}(\mathcal{A})$, and $l_i \in \mathbb{Z}$. The space of morphisms is given by

$$\mathrm{hom}_{\Sigma\mathcal{A}} \left(\bigoplus_{i \in I} c_i[l_i], \bigoplus_{j \in J} d_j[m_j] \right) = \bigoplus_{i,j} \mathrm{hom}_{\mathcal{A}}(c_i, d_j)[m_j - l_i],$$

with the obvious A_∞ -operations inherited from \mathcal{A} .

- (ii) A *twisted complex* over \mathcal{A} is a set $(\{c_i\}_{i \in I}, \{x_{ij}\}_{i,j \in I})$, where $I \subset \mathbb{Z}$ is a finite index set, c_i is an object of $\Sigma\mathcal{A}$, and x_{ij} is an element of $\mathrm{hom}_{\Sigma\mathcal{A}}^{i-j+1}(c_i, c_j)$ satisfying

$$\sum_{n=1}^{\infty} \sum_{l=l_0 < l_1 < \dots < l_n=m} \mathfrak{m}_n(x_{l_{n-1}l_n} \otimes \dots \otimes x_{l_0l_1}) = 0$$

for any $l, m \in \mathbb{Z}$. We assume $x_{ij} = 0$ for $i > j$.

Twisted complexes form an A_∞ -category:

Lemma 25. For an A_∞ -category \mathcal{A} , there is another A_∞ -category $\mathrm{Pre-Tr}(\mathcal{A})$ whose objects are twisted complexes of \mathcal{A} such that the space of morphisms between two twisted complexes $c = (\{c_i\}_i, \{x_{ij}\}_{i,j})$ and $d = (\{d_i\}_i, \{y_{ij}\}_{i,j})$ is

$$\bigoplus_{i,j} \mathrm{hom}_{\Sigma\mathcal{A}}(c_i[i], d_j[j]).$$

See, e.g., [23] for an explicit formula of the A_∞ -operations and the proof of the A_∞ -relations.

The cohomology category of $\mathrm{Pre-Tr}(\mathcal{A})$ is called the *bounded derived category* of \mathcal{A} and will be denoted by $D^b(\mathcal{A})$. It has a natural structure of a triangulated category by a straightforward adaptation of [4, proposition 2] to the A_∞ situation.

5.2 Fukaya category

For a pair of a symplectic manifold M and a family $\{L_i\}_i$ of at most countably many Lagrangian submanifolds, its Fukaya category is the A_∞ -category whose set of objects is $\{L_i\}_i$ and whose spaces of morphisms are the Lagrangian intersection Floer complexes. The Floer cohomology of two exact

Lagrangian submanifolds is defined by Floer [21]. The A_∞ -structure is introduced by Fukaya [22] and used by Kontsevich [34] to formulate his *homological mirror symmetry* conjecture. The task of defining it in full generality is undertaken by Fukaya, Oh, Ohta, and Ono [24], although we deal only with exact Lagrangian submanifolds in exact symplectic manifolds in this paper.

Definition 26. A symplectic manifold M with its symplectic form ω is exact if there is a one-form θ on M such that $\omega = d\theta$. A Lagrangian submanifold L of M is exact if there is a function ϕ on L such that $\theta|_L = d\phi$.

For two exact Lagrangian submanifolds L and L' intersecting transversally, their Floer complex $\text{hom}(L, L')$ is defined as the k -vector space spanned by their intersection points:

$$\text{hom}(L, L') = \bigoplus_{p \in L \cap L'} k[p].$$

When the intersection is not transversal but clean, we can find an exact symplectomorphism $\phi : M \rightarrow M$ so that L and $\phi(L')$ intersect transversally. The quasi-isomorphism class of the resulting $\text{hom}(L, \phi(L'))$ does not depend on the choice of ϕ due to the basic isotopy invariance property of the Lagrangian intersection Floer theory.

In general, the Floer complexes do not admit \mathbb{Z} -gradings. To equip them with \mathbb{Z} -gradings, we need the following concept of graded Lagrangian submanifolds, introduced by Kontsevich [34].

For a symplectic manifold M of dimension $2d$ with a compatible almost complex structure J such that $2c_1(M, J) \in H^2(M, \mathbb{Z})$ is zero, a *grading* of M is a choice of (a homotopy class of) a nowhere-vanishing section $\Omega^{\otimes 2}$ of $(\wedge^d(T^*M, J))^{\otimes 2}$, where (T^*M, J) is the holomorphic part of the cotangent bundle. For a Lagrangian submanifold L of M with a grading $\Omega^{\otimes 2}$, define a function $s_L : L \rightarrow S^1 = \mathbb{C}^\times / \mathbb{R}_{>0}$ by sending $x \in L$ to

$$s_L(x) = [\Omega^{\otimes 2}((e_1 \wedge \cdots \wedge e_d)^{\otimes 2})],$$

where $\{e_i\}_{i=1}^d$ is a basis of $T_x L$. A *grading* of L is a choice of a lift $\tilde{s}_L : L \rightarrow \mathbb{R}$ of s_L to the universal cover \mathbb{R} of S^1 . The pair $\tilde{L} = (L, \tilde{s}_L)$ of a Lagrangian submanifold L and its grading \tilde{s}_L is called a *graded Lagrangian submanifold*. For a graded Lagrangian submanifold $\tilde{L} = (L, \tilde{s}_L)$ and an integer $l \in \mathbb{Z}$, $\tilde{L}[l]$ will denote the graded Lagrangian submanifold $(L, \tilde{s}_L + l)$. This defines a free action of \mathbb{Z} on the set of gradings on L .

For an intersection point $p \in L \cap L'$ of two graded Lagrangian submanifolds \tilde{L} and \tilde{L}' , one can define its *Maslov index* $\mu(\tilde{L}, \tilde{L}'; p)$. The importance of Maslov indices lies in the fact that they appear in the index theorem for Cauchy-Riemann operators with Lagrangian boundary conditions, and hence in the virtual dimension of the moduli space of pseudoholomorphic

maps. See, e.g., [40] for the definition and basic properties of Maslov indices. For two transversal graded Lagrangian submanifolds \tilde{L} and \tilde{L}' , we can introduce a \mathbb{Z} -grading on the Floer complex $\text{hom}(L, L')$ so that the basis $[p] \in \text{hom}(L, L')$ for $p \in L \cap L'$ is homogeneous of degree $\mu(\tilde{L}, \tilde{L}'; p)$. We write $\text{hom}(\tilde{L}, \tilde{L}')$ for the Floer complex $\text{hom}(L, L')$ equipped with this grading.

The A_∞ -operations are defined as follows: Let $\tilde{L}_0, \dots, \tilde{L}_l$ be a sequence of mutually transversal graded Lagrangian submanifolds and $p_i \in L_i \cap L_{i+1}$, $i = 0, \dots, l$ be a collection of their intersection points. Here, $L_{l+1} = L_0$ by notation. Then the coefficient of p_l in $\mathfrak{m}_l(p_0, \dots, p_{l-1})$ is given by the virtual number of pseudoholomorphic maps $\varphi : (D^2, \vec{z}) \rightarrow M$ from an $(l+1)$ -pointed disk (D^2, \vec{z}) such that $\varphi(\partial_i D^2) \subset L_i$ and $\varphi(z_i) = p_i$. Here, an $(l+1)$ -pointed disk is a pair of a two-dimensional disk D^2 with the standard complex structure and $l+1$ points $\vec{z} = (z_0, \dots, z_l)$ on its boundary respecting the cyclic order, and $\partial_i D^2$ denotes the interval on the boundary of D^2 between z_{i-1} and z_i for $i = 0, \dots, l$.

5.3 Homological mirror symmetry for A_n -singularities

The mirrors of A_n -singularities were introduced by Hashiba and Naka [26], following the construction of Hori, Iqbal, and Vafa [27, 28]. Our treatment here follows Khovanov and Seidel [33] closely, which amounts to the homological mirror symmetry for \mathcal{C} . Introduce the affine manifold

$$W = \{(x, y, z) \in \mathbb{C}^2 \times \mathbb{C}^\times \mid xyz = z^{n+1} - 1\}$$

equipped with the exact symplectic structure

$$\omega = -\frac{\sqrt{-1}}{2} \left(dx \wedge d\bar{x} + dy \wedge d\bar{y} + \frac{dz \wedge d\bar{z}}{|z|^2} \right) \Big|_W.$$

The fiber of the projection

$$\begin{array}{ccc} \pi : & W & \rightarrow \mathbb{C}^\times \\ & \Downarrow & \Downarrow \\ & (x, y, z) & \mapsto z \end{array}$$

is a conic in \mathbb{C}^2 , which degenerates to the cone $xy = 0$ over the discriminant set $\Delta = \{\zeta^i\}_{i=0}^n$. Here, $\zeta = \exp[2\pi\sqrt{-1}/(n+1)]$ is the primitive $(n+1)$ th roots of unity.

A smooth map $c : [0, 1] \rightarrow \mathbb{C}^\times$ will be called an *admissible curve* if $c(0), c(1) \in \Delta$, $c(t) \notin \Delta$ for $0 < t < 1$, and c is injective. With an admissible curve c , we can associate a Lagrangian submanifold L_c of W by arranging the vanishing cycles of π along c :

$$L_c = \bigcup_{0 \leq t \leq 1} \{(x, y, c(t)) \in W \mid |x| = |y|\}.$$

Since L_c is diffeomorphic to a sphere, it is an exact Lagrangian submanifold.

Two admissible curves c and c' are called isotopic if there is a continuous path $\phi : [0, 1] \rightarrow \text{Diff}_0(\mathbb{C}^\times; \Delta)$ such that $\phi(0) = \text{id}$ and $\phi(1)(c) = c'$. We write this as $c \simeq c'$. In this case, $L_{\phi(t)}$ gives a smooth family of Lagrangian submanifolds of W connecting L_c and $L_{c'}$. Since L_c has the vanishing first cohomology, it follows that L_c and $L_{c'}$ are related by an exact symplectic isotopy.

Equip W with the grading given by the second tensor power $\Omega^{\otimes 2}$ of the holomorphic volume form

$$\Omega = \text{Res} \frac{dx \wedge dy \wedge dz}{xyz - z^{n+1} + 1} = \frac{dy \wedge dz}{yz}.$$

For an admissible curve c , the Lagrangian submanifold L_c of W has an S^1 -action

$$\begin{aligned} S^1 \ni e^{\sqrt{-1}\theta} & : \quad \begin{array}{c} L_c \\ \Downarrow \\ (x, y, z) \end{array} \rightarrow \begin{array}{c} L_c \\ \Downarrow \\ (e^{\sqrt{-1}\theta}x, e^{-\sqrt{-1}\theta}y, z) \end{array}. \end{aligned}$$

The orbit space of this S^1 -action can be identified with the image $c([0, 1]) = \pi(L_c) \subset \mathbb{C}^\times$, and for $t \in (0, 1)$, the tangent space at a point $(x, y, c(t)) \in L_c$ in $\pi^{-1}(c(t))$ is spanned by $\frac{\partial}{\partial t} = c'(t)\frac{\partial}{\partial z} + \star \frac{\partial}{\partial y}$ and $\frac{\partial}{\partial \theta} = \sqrt{-1}y\frac{\partial}{\partial y}$. Here, we have chosen $\frac{\partial}{\partial y}$ and $\frac{\partial}{\partial z}$ as a basis of the tangent space of W , and \star is a term which is irrelevant for the following calculation of s_{L_c} . Then

$$s_{L_c}(x, y, c(t)) = \left[\frac{dy \wedge dz}{yc(t)} \left(c'(t)\frac{\partial}{\partial z} \wedge \sqrt{-1}y\frac{\partial}{\partial y} \right) \right]^2 = [-c'(t)^2 c(t)^{-2}].$$

Hence the grading of L_c admits the following description: A curve $c : [0, 1] \rightarrow \mathbb{C}^\times$ defines a map $s_c : [0, 1] \rightarrow S^1 \cong \mathbb{C}^\times / \mathbb{R}_{>0}$ by $s_c(t) = [-c'(t)^2 c(t)^{-2}]$, and a *grading* of c is a lift $\tilde{s}_c : [0, 1] \rightarrow \mathbb{R}$ of s_c to the universal cover \mathbb{R} of S^1 . Since

$$s_{L_c}(x, y, c(t)) = s_c(t)$$

for $(x, y, c(t)) \in L_c$, a grading of c is in one-to-one correspondence with a grading of L_c . For a graded curve \tilde{c} , \tilde{L}_c will denote the corresponding graded Lagrangian submanifold of W . Any admissible curve admits a grading, and there is a free and transitive action of \mathbb{Z} on the set of gradings on c .

For a transversal intersection point p of two graded curves $\tilde{c}_0 = (c_0, \tilde{s}_{c_0})$ and $\tilde{c}_1 = (c_1, \tilde{s}_{c_1})$, define an integer $\mu(\tilde{c}_0, \tilde{c}_1; p)$ as follows: Fix a small circle l around the point p and take an intersection point $\alpha_0 = c_0(t_0)$ of l and c_0 . Let $\alpha_1 = c_1(t_1)$ be the first intersection point of l and c_1 as one goes from α_0 along l clockwise. Take the clockwise arc $\alpha : [0, 1] \rightarrow l$ from α_0 to α_1 and a smooth map $\pi : [0, 1] \rightarrow S^1 \cong \mathbb{C}^\times / \mathbb{R}_{>0}$ such that $\pi(0) = [-c'_0(t_0)^2 c_0(t_0)^{-2}]$, $\pi(1) = [-c'_1(t_1)^2 c_1(t_1)^{-2}]$, and $\pi(t) \neq [-\alpha'(t)^2 \alpha(t)^{-2}]$ for all t . Then there

is a unique lift $\tilde{\pi} : [0, 1] \rightarrow \mathbb{R}$ of π such that $\tilde{\pi}(0) = \tilde{s}_{c_0}(t_0)$, and $\mu(\tilde{c}_0, \tilde{c}_1; p)$ is defined by

$$\mu(\tilde{c}_0, \tilde{c}_1; p) = \tilde{s}_{c_1}(t_1) - \tilde{\pi}(1).$$

Recall that two curves c_0 and c_1 are said to intersect *minimally* if they intersect transversally and satisfy the following condition: Take any two points $z_- \neq z_+$ in $c_0 \cap c_1$ which do not both lie in Δ , and two arcs $\alpha_0 \subset c_0$, $\alpha_1 \subset c_1$ with endpoints z_- , z_+ , such that $\alpha_0 \cap \alpha_1 = \{z_-, z_+\}$. Let K be the connected component of $\mathbb{C}^\times \setminus (c_0 \cup c_1)$ which is bounded by $\alpha_0 \cup \alpha_1$. Then if K is topologically an open disk, it must contain at least one point of Δ .

For two graded curves \tilde{c}_0 and \tilde{c}_1 intersecting minimally, their *graded intersection number* is defined by

$$I^{\text{gr}}(\tilde{c}_0, \tilde{c}_1) = (1 + q) \sum_{p \in (c_0 \cap c_1) \setminus \Delta} q^{\mu(c_0, c_1; p)} + \sum_{p \in c_0 \cap c_1 \cap \Delta} q^{\mu(c_0, c_1; p)}.$$

Here, the factor of $1 + q$ in front of the sum over $(c_0 \cap c_1) \setminus \Delta$ comes from the Poincaré polynomial $\sum_r q^r \dim_k H^r(S^1; k) = 1 + q$ of S^1 and the fact that L_{c_0} and L_{c_1} intersect cleanly along S^1 over $p \in (c_0 \cap c_1) \setminus \Delta$.

The following result is due to Khovanov and Seidel:

Theorem 27 ([33, lemma 6.19]). *Let \tilde{L}_0 and \tilde{L}_1 be the exact graded Lagrangian two-spheres in W coming from two admissible graded curves \tilde{c}_0 and \tilde{c}_1 on the z -plane. Assume that c_0 and c_1 intersect transversally and minimally. Then the Poincaré polynomial of the Floer cohomology groups of \tilde{L}_0 and \tilde{L}_1 is given by the graded intersection number of \tilde{c}_0 and \tilde{c}_1 :*

$$\sum_{r \in \mathbb{Z}} q^r \dim_k H^r(\text{hom}(\tilde{L}_0, \tilde{L}_1), \mathfrak{m}_1) = I^{\text{gr}}(\tilde{c}_0, \tilde{c}_1).$$

Since $I^{\text{gr}}(\tilde{c}_0, \tilde{c}_1)$ at $q = 1$ does not depend on the choice of the grading, we write

$$I(c_0, c_1) = \frac{1}{2} I^{\text{gr}}(\tilde{c}_0, \tilde{c}_1)|_{q=1}$$

and call it the *geometric intersection number*.

For $i = 0, \dots, n$, let $c_i : [0, 1] \rightarrow \mathbb{C}^\times$ be the straight line segment from ζ^i to ζ^{i+1} as in Figure 1. Equip c_i with the grading such that the value of \tilde{s}_{c_i} at the midpoint $(\zeta_i + \zeta_{i+1})/2$ is the same for all $i = 0, \dots, n$, and let \tilde{L}_i be the corresponding graded Lagrangian submanifold of W . Let further $\mathfrak{Fuk} W$ be the Fukaya category of W whose set of objects is $\{\tilde{L}_i\}_{i=0}^n$.

The following theorem is the homological mirror symmetry for A_n -singularities. Note that we do not need the assumption on the characteristic of k in the proof.

Theorem 28. *There is an equivalence*

$$\mathcal{D}_k \cong D^b \mathfrak{Fuk} W$$

of triangulated categories.

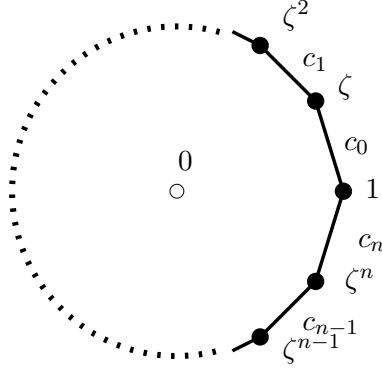


Figure 1: The curves c_i 's

Proof. We can see that there are no non-constant holomorphic maps from a disk to W with $\{L_i\}_{i=0}^n$ as Lagrangian boundary conditions: If $\phi : D^2 \rightarrow W$ is a holomorphic map from a disk such that $\phi(\partial D^2) \subset \bigcup_{i=0}^n L_i$, then $\pi \circ \phi$ is a holomorphic map from D^2 to \mathbb{C}^\times with the boundary condition $\phi(\partial D^2) \subset \bigcup_{i=0}^n c_i$. Since non-constant holomorphic maps are open and there are no continuous open maps from D^2 to \mathbb{C}^\times with the above boundary condition, $\pi \circ \phi$ must be constant. Hence ϕ is a holomorphic map from D^2 into the fiber $\pi^{-1}(z)$ for some $z \in \bigcup_{i=0}^n c_i \subset \mathbb{C}^\times$. It is obvious that for any $z \in \mathbb{C}^\times$, there are no non-constant holomorphic maps $\phi : D^2 \rightarrow \pi^{-1}(z)$ such that

$$\phi(\partial D^2) \subset \{(x, y, z) \in \pi^{-1}(z) \mid |x| = |y|\}.$$

Therefore the structure of the Fukaya category $\mathfrak{Fuk} W$ is easy to describe: The spaces of morphisms are given by

$$\mathrm{hom}_{\mathfrak{Fuk} W}(\tilde{L}_i, \tilde{L}_j) = \begin{cases} k \cdot e_i \oplus k \cdot f_i & \text{if } i = j, \\ k \cdot x_i & \text{if } j = i + 1, \\ k \cdot y_i & \text{if } j = i - 1, \\ 0 & \text{otherwise,} \end{cases}$$

where $\deg(e_i) = 0$, $\deg(x_i) = \deg(y_i) = 1$, and $\deg(f_i) = 2$. The A_∞ -operations are trivial except for \mathfrak{m}_2 , and e_i is the identity morphism of \tilde{L}_i for $i = 0, \dots, n$. Nontrivial \mathfrak{m}_2 are given by

$$\mathfrak{m}_2(y_{i+1}, x_i) = \mathfrak{m}_2(x_{i-1}, y_i) = f_i, \quad i = 0, \dots, n.$$

Hence the total morphism algebra $\bigoplus_{i,j=0}^n \mathrm{hom}_{\mathfrak{Fuk} W}(\tilde{L}_i, \tilde{L}_j)$ is not only an A_∞ -algebra but a differential graded algebra (i.e., $\mathfrak{m}_n = 0$ for $n \geq 3$) with the trivial differential (i.e., $\mathfrak{m}_1 = 0$). Note that for any $i, j = 0, \dots, n$, we have

$$\mathrm{hom}_{\mathfrak{Fuk} W}(\tilde{L}_i, \tilde{L}_j) = \mathrm{Ext}_{\mathcal{A}_n^{(1)}}^*(S_i, S_j),$$

where $\mathcal{A}_n^{(1)}$ is the path algebra with relations appearing in §3 and S_i is the simple $\mathcal{A}_n^{(1)}$ -module corresponding to the idempotent $(i) \in \mathcal{A}_n^{(1)}$. Since the abelian category $\text{mod } \mathcal{A}_n^{(1)}$ of finitely generated right $\mathcal{A}_n^{(1)}$ -modules has enough injectives, its derived category $D^b \text{mod } \mathcal{A}_n^{(1)}$ has an enhancement in the sense of Bondal and Kapranov [4]. Since $\mathcal{A}_n^{(1)}$ is Koszul over the semisimple ring $k^{\oplus(n+1)}$, the total endomorphisms DG -algebra of $\bigoplus_{i=0}^n S_i$ in the enhancement of $D^b \text{mod } \mathcal{A}_n^{(1)}$ is formal. Then a theorem of Bondal and Kapranov [4, theorem 1] shows that $D^b \mathfrak{Fuk} W$ is equivalent as a triangulated category to the smallest triangulated subcategory of $D^b \text{mod } \mathcal{A}_n^{(1)}$ containing $\{S_i\}_{i=0}^n$. This subcategory is equivalent to the bounded derived category $D^b \text{mod}_0 \mathcal{A}_n^{(1)}$ of finitely-generated nilpotent $\mathcal{A}_n^{(1)}$ -modules. Since $D^b \text{mod}_0 \mathcal{A}_n^{(1)}$ is triangle-equivalent to \mathcal{D}_k by the McKay correspondence, we obtain an equivalence $\mathcal{D}_k \cong D^b \mathfrak{Fuk} W$ as desired. \square

Now we discuss the symplectic Dehn twist. Let

$$T = \{(a_0, \dots, a_n) \in \mathbb{C}^{n+1} \mid a_0 \cdots a_n = 1\}$$

be an algebraic torus of dimension n and

$$D = \{(a_0, \dots, a_n) \in T \mid a_i = a_j \text{ for some } i \neq j\}$$

be its big diagonal. The symmetric group \mathfrak{S}_{n+1} of rank $n+1$ acts on T by permutations, and let S be the quotient $(T \setminus D)/\mathfrak{S}_{n+1}$. Consider the family

$$\mathcal{W} = \{(x, y, z, a) \in \mathbb{C}^2 \times \mathbb{C}^\times \times S \mid xyz = (z - a_0) \cdots (z - a_n)\} \rightarrow S$$

of exact symplectic manifolds over S , equipped with the relative grading given by the relative holomorphic volume form $\Omega = dy/y \wedge dz/z$. Since any fibration of exact symplectic manifolds is locally trivial by Moser [35], a loop in the parameter space S induces a graded symplectomorphism of the fiber as its monodromy. For $i = 0, \dots, n$, let τ_i be the monodromy along the loop around $a_i = a_{i+1}$ as in Figure 2, starting and ending at $[(a_0, a_1, \dots, a_n)] = [(1, \zeta, \dots, \zeta^n)] \in S$. Then τ_i acts both on graded curves

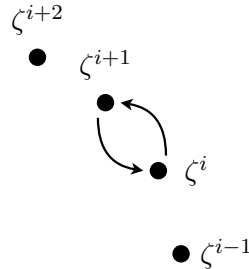


Figure 2: The Dehn twist τ_i

and on graded Lagrangian submanifolds, and the construction of graded Lagrangian submanifolds from graded curves commutes with this action; for a graded curve \tilde{c} , we have

$$\tau_i(\tilde{L}_{\tilde{c}}) = \tilde{L}_{\tau_i(\tilde{c})}.$$

Let $\mathfrak{Fuk} W^\sim$ be the Fukaya category whose set of objects is the images of L_i , $i = 0, \dots, n$ by the compositions of τ_j for $j = 0, \dots, n$. The following theorem is due to Seidel:

Theorem 29 ([39, proposition 9.1]). *The graded Lagrangian submanifold $\tau_i(\tilde{L})$ is isomorphic to the twisted complex $T_{\tilde{L}_i}(\tilde{L})$ in $D^b \mathfrak{Fuk} W^\sim$.*

The above theorem also shows that $D^b \mathfrak{Fuk} W^\sim$ is equivalent to $D^b \mathfrak{Fuk} W$ as a triangulated category.

5.4 The proof of the faithfulness in characteristic two

Here we prove the faithfulness of the affine braid group action on $D^b \mathfrak{Fuk} W^\sim$ given by

$$\begin{array}{ccc} \rho & : & B_n^{(1)} \rightarrow \text{Auteq } D^b \mathfrak{Fuk} W^\sim \\ & \Downarrow & \Downarrow \\ & \sigma_i & \mapsto \tau_i. \end{array}$$

Since the action of $B_n^{(1)}$ on graded curves and graded Lagrangian submanifolds commute, we can work with graded curves on \mathbb{C}^\times instead of graded Lagrangian submanifolds of W .

The following lemma is an affine version of [33, lemma 3.6]:

Lemma 30. *If $b \in B_n^{(1)}$ satisfies*

$$I(b(c_i), b'(c_j)) = I(c_i, b'(c_j)) \tag{6}$$

for any $i, j = 0, \dots, n$ and any $b' \in B_n^{(1)}$, then b is the identity.

Proof. Think of $B_n^{(1)}$ as a subgroup of $\pi_0(\text{Diff}_0(\mathbb{C}^\times; \Delta))$ and take an element b as above. We prove that $b(c_i) \simeq c_i$ for $i = 0, \dots, n$, which implies $b = \text{id}$. It suffices to prove $b(c_0) \simeq c_0$ by symmetry. Put $c = b(c_0)$. By substituting $b' = \text{id}$, $i = 0$, and $j = 1, \dots, n$ into (6), we obtain

$$I(c, c_1) = I(c, c_n) = \frac{1}{2}$$

and

$$I(c, c_2) = \dots = I(c, c_{n-1}) = 0.$$

This leaves two possibilities c' and c'' in Figure 3 for the homotopy class of c other than c_0 . However, we have

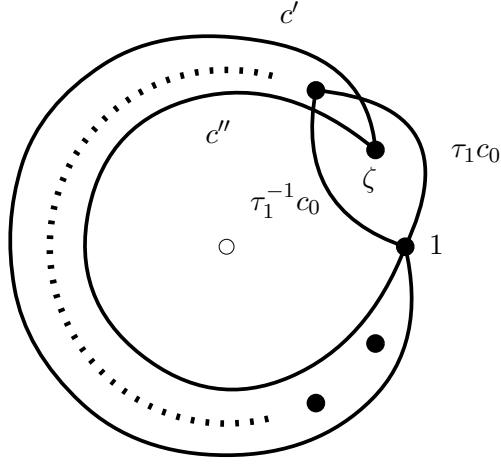


Figure 3: c' and c''

$$I(c', \tau_1 c_0) = 1 \neq \frac{1}{2} = I(c_0, \tau_1 c_0)$$

and

$$I(c'', \tau_1^{-1} c_0) = 1 \neq \frac{1}{2} = I(c_0, \tau_1^{-1} c_0),$$

which shows that neither c' nor c'' satisfies (6) for $b' = \sigma_1$ and $b' = \sigma_1^{-1}$ respectively. Hence we have $c \simeq c_0$. \square

Now we can prove the faithfulness in characteristic two:

Theorem 31. *If $b \in B_n^{(1)}$ satisfies $\rho(b) = \text{id}$ in $\text{Auteq } D^b \mathfrak{Fut} W^\sim$, then $b = \text{id}$ in $B_n^{(1)}$.*

Proof. Since $\rho(b) = \text{id}$,

$$\dim \text{Hom}^*(\rho(b)(L_i), \rho(b')(L_j)) = \dim \text{Hom}^*(L_i, \rho(b')(L_j))$$

for any $i, j = 0, \dots, n$ and any $b' \in B_n^{(1)}$. Since

$$\dim \text{Hom}^*(\rho(b)(L_i), \rho(b')(L_j)) = 2I(b(c_i), b'(c_j)),$$

by Theorem 27, we obtain $b = \text{id}$ from Lemma 30. \square

6 Lifting to any characteristic

In this section, we lift the above result to any characteristic. For that purpose, we consider

$$Y_{\mathbb{Z}} = \text{Spec } \mathbb{Z}[x, y, z]/(xy + z^{n+1})$$

and the minimal resolution $f : X_{\mathbb{Z}} \rightarrow Y_{\mathbb{Z}}$ with the exceptional locus $Z_{\mathbb{Z}} = \cup_{i=1}^n C_{i,\mathbb{Z}}$. For any noetherian ring R , we denote $X_R = X_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$, $Y_R = Y_{\mathbb{Z}} \otimes_{\mathbb{Z}} R$, $\mathcal{D}_R = D^b \text{coh}_{Z_R} X_R$, etc.

We say that an object $\mathcal{E} \in \mathcal{D}_R$ is spherical if

- (i) \mathcal{E} is quasi-isomorphic to a bounded complex of R -flat coherent sheaves on X_R whose cohomology sheaves are supported at Z_R ,
- (ii) $\text{Ext}_{X_R}^0(\mathcal{E}, \mathcal{E}) \cong R \cong \text{Ext}_{X_R}^2(\mathcal{E}, \mathcal{E})$,
- (iii) $\text{Ext}_{X_R}^i(\mathcal{E}, \mathcal{E}) = 0$ for $i \neq 0, 2$.

Remark 32. (i) Since $X_{\mathbb{Z}}$ is smooth over \mathbb{Z} , we can replace “ R -flat” in the first condition by “locally free.”

(ii) When R is regular, then the first condition is automatically satisfied.

For an object $\mathcal{E} \in \mathcal{D}_R$ and any noetherian R -algebra S , we put $\mathcal{E}_S := \mathcal{E} \otimes_R^{\mathbb{L}} S$.

Lemma 33. *If $\mathcal{E} \in \mathcal{D}_R$ is spherical, so is \mathcal{E}_S .*

Proof. This follows from the base change theorem. (See [25, §7.7 and §7.8].) \square

Lemma 34. *Let (R, m) be a noetherian local ring with residue field k' . Suppose that $\mathcal{E}_0 \in \mathcal{D}_{k'}$ is a spherical object. Then there is a spherical object $\mathcal{E} \in \mathcal{D}_R$ with R -flat cohomology sheaves $\mathcal{H}^i(\mathcal{E})$ which satisfies $\mathcal{E}_{k'} \cong \mathcal{E}_0$.*

Proof. Put $\mathcal{H}_0^i := \mathcal{H}^i(\mathcal{E}_0)$. Then, as in [30, §4.1 and §4.2], \mathcal{E}_0 determines elements $e^i(\mathcal{E}_0) \in \text{Ext}_{X_{k'}}^2(\mathcal{H}_0^i, \mathcal{H}_0^{i-1})$ and the isomorphism class of \mathcal{E}_0 is determined by these data. We construct \mathcal{E} by lifting these data to X_R . \mathcal{H}_0^i is a direct sum of sheaves that are line bundles on subchains of exceptional curves and hence can be lifted to a sheaf $\mathcal{H}^i \in \text{coh}_{Z_R} X_R$ flat over R . Since $\text{Ext}_{X_{k'}}^p(\mathcal{H}_0^i, \mathcal{H}_0^j) = 0$ for $p \neq 0, 2$ by [30, proposition 4.5], the base change theorem implies that $\text{Ext}_{X_R}^p(\mathcal{H}^i, \mathcal{H}^j)$ is R -free and $\text{Ext}_{X_{k'}}^p(\mathcal{H}_0^i, \mathcal{H}_0^j) \cong \text{Ext}_{X_R}^p(\mathcal{H}^i, \mathcal{H}^j) \otimes_R k'$. Especially, $e^i(\mathcal{E}_0)$ can be lifted to an element $e^i \in \text{Ext}_{X_R}^2(\mathcal{H}^i, \mathcal{H}^{i-1})$. As in [30, proposition 4.2], it follows from $\text{Ext}_{X_R}^{\geq 3}(\mathcal{H}^i, \mathcal{H}^j) = 0$ that there is an object $\mathcal{E} \in \mathcal{D}_R$ with $\mathcal{H}^i(\mathcal{E}) \cong \mathcal{H}^i$ and $e^i(\mathcal{E}) = e^i$, which is unique up to isomorphism. Then the construction yields $\mathcal{E}_{k'} \cong \mathcal{E}_0$ and therefore \mathcal{E} is spherical by the base change theorem. \square

Corollary 35. *Let $\mathcal{E} \in \mathcal{D}_R$ be spherical for some noetherian ring R . Then the cohomology sheaves $\mathcal{H}^i(\mathcal{E})$ are R -flat.*

Proof. We may assume R is local with residue field k' . By the previous lemma, there is a spherical object $\mathcal{F} \in \mathcal{D}_R$ with R -flat cohomology sheaves satisfying $\mathcal{F}_{k'} \cong \mathcal{E}_{k'}$. Since we have $\mathrm{Ext}_{X_{k'}}^1(\mathcal{E}_{k'}, \mathcal{E}_{k'}) = 0$ and $\mathrm{Ext}_{X_{k'}}^0(\mathcal{E}_{k'}, \mathcal{E}_{k'}) \cong k'$, the deformation theory in [29] implies that \mathcal{E} and \mathcal{F} are isomorphic on some étale covering of $\mathrm{Spec} R$. This proves that $\mathcal{H}^i(\mathcal{E})$ is R -flat. \square

For a spherical object $\mathcal{E} \in \mathcal{D}_R$, we can define the twist functor $T_{\mathcal{E}}$ as a Fourier-Mukai transform with respect to $\{\mathcal{E}^\vee \boxtimes \mathcal{E} \rightarrow \mathcal{O}_\Delta\}$. Then, for any R -algebra S , the base change theorem (see [6, §2.1]) implies that the twist functors $T_{\mathcal{E}}$ and $T_{\mathcal{E}_S}$ commute with the functor $\cdot \otimes_R^{\mathbb{L}} S : \mathcal{D}_R \rightarrow \mathcal{D}_S$. In particular, for a spherical object \mathcal{E} in $\mathcal{D}_{\mathbb{Z}}$, we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{D}_k & \longleftarrow & \mathcal{D}_{\mathbb{Z}} & \longrightarrow & \mathcal{D}_{\mathbb{F}_2} \\ T_{\mathcal{E}_k} \downarrow & & T_{\mathcal{E}} \downarrow & & T_{\mathcal{E}_{\mathbb{F}_2}} \downarrow \\ \mathcal{D}_k & \longleftarrow & \mathcal{D}_{\mathbb{Z}} & \longrightarrow & \mathcal{D}_{\mathbb{F}_2} \end{array}$$

Recall that k is the base field of any characteristic. Thus for an element b of the affine braid group, we have autoequivalences $\rho_k(b)$, $\rho_{\mathbb{Z}}(b)$, $\rho_{\mathbb{F}_2}(b)$ of \mathcal{D}_k , $\mathcal{D}_{\mathbb{Z}}$, $\mathcal{D}_{\mathbb{F}_2}$, respectively, which commute with $\cdot \otimes_{\mathbb{Z}}^{\mathbb{L}} k$ and $\cdot \otimes_{\mathbb{Z}}^{\mathbb{L}} \mathbb{F}_2$.

Proposition 36. *If $\rho_k(b)$ is isomorphic to the identity, then $\rho_{\mathbb{F}_2}(b)$ is also isomorphic to the identity.*

Proof. Since $\rho_k(b)$ is the identity, we have $\rho_{\mathbb{Z}}(b)(\mathcal{O}_{C_{i,\mathbb{Z}}}(d)) \otimes k \cong \mathcal{O}_{C_{i,k}}(d)$ for any $d \in \mathbb{Z}$. Now $\rho_{\mathbb{Z}}(b)(\mathcal{O}_{C_{i,\mathbb{Z}}}(d))$ has flat cohomologies by Corollary 35 and hence we have $\rho_{\mathbb{Z}}(b)(\mathcal{O}_{C_{i,\mathbb{Z}}}(d)) \cong \mathcal{O}_{C_{i,\mathbb{Z}}}(d)$. This implies $\rho_{\mathbb{F}_2}(b)(\mathcal{O}_{C_{i,\mathbb{F}_2}}(d)) \cong \mathcal{O}_{C_{i,\mathbb{F}_2}}(d)$. Therefore $\rho_{\mathbb{F}_2}(b)$ is a Fourier-Mukai functor which sends the structure sheaf of a closed point to the structure sheaf of a closed point. Since $\rho_{\mathbb{F}_2}(b)$ acts as the identity functor on objects supported outside the exceptional set, $\rho_{\mathbb{F}_2}(b)$ is isomorphic to a functor of the form $\cdot \otimes \mathcal{L}$ for some line bundle \mathcal{L} on X . The fact that $\rho_{\mathbb{F}_2}(b)(\mathcal{O}_{C_{i,\mathbb{F}_2}}(d)) \cong \mathcal{O}_{C_{i,\mathbb{F}_2}}(d)$ for any d shows that \mathcal{L} must be trivial. \square

By combining Theorems 28, 29, 31 and Proposition 36, we obtain the following:

Corollary 37. *The homomorphism ρ_k is injective for any field k .*

7 Connectedness of $\mathrm{Stab} \mathcal{C}$

In this section, we prove the connectedness of $\mathrm{Stab} \mathcal{C}$. Our strategy is the following:

- (i) First we prove Lemma 38, which states that any spherical object of \mathcal{C} can be obtained from $\mathcal{O}_{C_i}(-1)$ for some $i = 1, \dots, n$ by the action of $\text{Br}(\mathcal{C})$. This shows, on the mirror side, that any spherical object can be represented not only by a twisted complex of graded Lagrangian submanifolds but by an honest graded Lagrangian submanifold of W obtained by twisting L_1, \dots, L_n along themselves. This opens up a way to use topological arguments on configurations of curves on a disk to tackle the problem of the connectedness of $\text{Stab } \mathcal{C}$.
- (ii) Let Σ be a connected component of $\text{Stab } \mathcal{C}$. By Lemma 15, we can find a stability condition $\sigma = (Z, \mathcal{P}) \in \Sigma$ such that $Z(E) \notin \mathbb{R}$ for any spherical object $E \in \mathcal{C}$.
- (iii) Let $\{E_1, \dots, E_m\}$ be the set of simple objects of the heart $\mathcal{P}((0, 1])$ of the above stability condition. Then $m = n$ and $\{E_1, \dots, E_n\}$ forms an A_n -configuration with a suitable order.
- (iv) The homological mirror symmetry allows us to use a topological argument on a configuration of curves on a disk to find an autoequivalence $\Phi \in \text{Br}(\mathcal{C})$ which brings E_i to the standard generator $\mathcal{O}_{C_i}(-1)$ simultaneously for all $i = 1, \dots, n$ in characteristic two. Then the lifting argument in §6 allows us to prove it in any characteristic.
- (v) The above autoequivalence Φ brings σ to $\Phi\sigma$, which belongs to the distinguished connected component of $\text{Stab } \mathcal{C}$ studied by Bridgeland. Since the action of $\text{Br}(\mathcal{C})$ preserves this connected component [11, theorem 1.1], $\text{Stab } \mathcal{C}$ is connected.

7.1 Normalizing spherical objects

The following lemma follows from the result in [30]:

Lemma 38. *Let E be a spherical object in \mathcal{C} . Then there is an autoequivalence $\Phi \in \text{Br}(\mathcal{C})$ such that $\Phi(E) \cong \mathcal{O}_{C_i}(-1)$ for some i .*

Proof. First note the fact that $E \in \mathcal{C}$ if and only if $\mathcal{H}^j(E) \in \mathcal{C}$ for any j by [8, lemma 3.1]. Combining this with the proof of Proposition 1.6 in [30], we can show that for a spherical object $E \in \mathcal{C}$ with $l(E) > 1$, there is an autoequivalence $\Phi \in \text{Br}(\mathcal{C})$ such that $l(E) > l(\Phi(E))$. Here, the length $l(E)$ of an object $E \in \mathcal{D}$ is defined in [30] as

$$l(E) = \sum_{i,p} \text{length}_{\mathcal{O}_{X,\eta_i}} \mathcal{H}^p(E)_{\eta_i},$$

where η_i is the generic point of C_i . We can check that the spherical object $E \in \mathcal{C}$ with $l(E) = 1$ has the form $\mathcal{O}_{C_i}(-1)[d]$ for some $i, d \in \mathbb{Z}$. Hence the equality $T_{\mathcal{O}_{C_i}(-1)}(\mathcal{O}_{C_i}(-1)) = \mathcal{O}_{C_i}(-1)[-1]$ completes the proof. \square

7.2 A_n -configurations from stability conditions

Fix a connected component Σ of $\text{Stab } \mathcal{C}$. We can show $V(\Sigma) = \text{Hom}(K(\mathcal{C}), \mathbb{C})$ in just the same way as in the proof of Lemma 15. Hence we have a stability condition $\sigma = (Z, \mathcal{P}) \in \Sigma$ such that $Z(E) \notin \mathbb{R}$ for any spherical object $E \in \mathcal{C}$. Then $\Im m Z$ determines a Weyl chamber and the corresponding simple root basis. We prove that for such a stability condition, the set of all simple objects in its heart $\mathcal{P}((0, 1])$ forms an A_n -configuration whose image in $K(\mathcal{C})$ is the simple root basis.

Lemma 39. *Let $\sigma = (Z, \mathcal{P}) \in \Sigma$ be a stability condition such that $Z(E) \notin \mathbb{R}$ for any spherical object $E \in \mathcal{C}$. Then the set of all simple objects in $\mathcal{P}((0, 1])$ consists of n mutually non-isomorphic elements. Moreover, this set forms an A_n -configuration of spherical objects in an appropriate order.*

Proof. Note that a simple object in $\mathcal{P}((0, 1])$ is always stable and any stable object in \mathcal{C} is spherical by Lemma 6. Moreover, the class of a stable object in $\mathcal{P}((0, 1])$ is a positive root. Thus $\mathcal{P}((0, 1])$ is a finite length category.

If two simple objects $E_1, E_2 \in \mathcal{P}((0, 1])$ determine the same class in $K(\mathcal{C})$, then $\chi(E_1, E_2) = 2$ holds and we have $E_1 \cong E_2$. Hence, the set of simple objects in $\mathcal{P}((0, 1])$ injects into the set of positive roots; in particular, it is finite. Let $\{E_1, \dots, E_m\}$ be the set of all simple objects in $\mathcal{P}((0, 1])$. Then $\mathcal{P}((0, 1])$ is the smallest extension closed subcategory of \mathcal{C} containing this set, since it is of finite length.

If $i \neq j$, then we have $\text{Hom}_{\mathcal{C}}^{\leq 0}(E_i, E_j) = 0$ since E_i and E_j are mutually non-isomorphic simple objects of the heart. Then the Serre duality implies $\text{Hom}_{\mathcal{C}}^{\geq 2}(E_i, E_j) = 0$. Hence we have

$$\chi(E_i, E_j) = -\dim \text{Hom}_{\mathcal{C}}^1(E_i, E_j) \leq 0. \quad (7)$$

First we show that $\{E_1, \dots, E_m\}$ is a linear basis in $K(\mathcal{C}) \otimes \mathbb{Q}$. Assume that we have an equality $\sum_i a_i [E_i] = 0$ for some $a_i \in \mathbb{Q}$. Put

$$\alpha = \sum_{a_i > 0} a_i [E_i] = - \sum_{a_j < 0} a_j [E_j].$$

Then (7) implies

$$\chi(\alpha, \alpha) = - \sum_{a_i > 0, a_j < 0} a_i a_j \chi(E_i, E_j) \leq 0,$$

and we obtain $\alpha = 0$. Since $[E_i]$'s are positive roots, we obtain $a_i = 0$ for all i . Thus they form a basis of $K(\mathcal{C}) \otimes \mathbb{Q}$. In particular we have $m = n$.

To show that $\{E_1, \dots, E_m\}$ forms a simple root basis of $K(\mathcal{C})$, let $\alpha \in K(\mathcal{C})$ be any root and write $\alpha = \sum_{i=1}^n a_i [E_i]$. Put $\alpha_1 = \sum_{a_i > 0} a_i [E_i]$ and $\alpha_2 = \sum_{a_j < 0} a_j [E_j]$. Then we can use (7) to see

$$\chi(\alpha, \alpha) = \chi(\alpha_1, \alpha_1) + \chi(\alpha_2, \alpha_2) + 2\chi(\alpha_1, \alpha_2) \geq \chi(\alpha_1, \alpha_1) + \chi(\alpha_2, \alpha_2).$$

Since α is a root, either α_1 or α_2 is zero. Hence $\{E_1, \dots, E_m\}$ forms a simple root basis. Consequently the set $\{E_1, \dots, E_n\}$ with a suitable order forms an A_n -configuration of spherical objects. \square

7.3 Normalizing an A_n -configuration

Here we prove the following lemma:

Lemma 40. *Let (E_1, \dots, E_n) be an A_n -configuration of spherical objects in \mathcal{C}_k , and assume that for any i , there is an element $\Phi \in \text{Br}(\mathcal{C}_k)$ such that $E_i \cong \Phi(\mathcal{O}_{C_j}(-1))$ for some j . Then there is an element $\Psi \in \text{Br}(\mathcal{C}_k)$ such that $\Psi(E_i) \cong \mathcal{O}_{C_i}(-1)$ for all $i = 1, \dots, n$.*

Proof. We use the mirror of \mathcal{C} introduced by Khovanov, Seidel, and Thomas [33, 41]. It is defined as the deformation

$$V = \{(x, y, z) \in \mathbb{C}^3 \mid xy = z(z-1) \cdots (z-n)\}$$

of the A_n -singularity, equipped with the symplectic form given by restricting the standard Euclidean Kähler form on \mathbb{C}^3 . The grading of V is given by the holomorphic volume form $\Omega = dy/y \wedge dz$. There are n graded Lagrangian submanifolds $\{\tilde{L}_i\}_{i=1}^n$ of V , where L_i is the Lagrangian submanifold coming from the line segment c_i from $i-1$ to i on the z -plane using the conic fibration

$$\begin{array}{ccc} \pi : & V & \rightarrow \mathbb{C} \\ & \Psi & \Psi \\ & (x, y, z) & \mapsto z \end{array}$$

in just the same way as for W . Let $\mathfrak{Fuk} V$ be the Fukaya category of V whose set of objects is $\{\tilde{L}_i\}_{i=1}^n$. Since $(\tilde{L}_i)_{i=1}^n$ is an A_n -configuration generating $D^b \mathfrak{Fuk} V$, $D^b \mathfrak{Fuk} V$ is equivalent to \mathcal{C} by the intrinsic formality of Seidel and Thomas [41, lemma 4.21]. We fix this equivalence by identifying $\mathcal{O}_{C_i}(-1)$ with \tilde{L}_i for $i = 1, \dots, n$.

Now we prove Lemma 40 in characteristic two. Then the lifting argument in §6 shows that it holds in any characteristic. The assumption $E_i \cong \Phi(\tilde{L}_j)$ shows that $E_i \in D^b \mathfrak{Fuk} V$ is isomorphic not only to a twisted complex of graded Lagrangian submanifolds but to an honest graded Lagrangian submanifold constructed from a graded curve \tilde{d}_i in \mathbb{C} whose endpoints lie in $\Delta = \{0, \dots, n\}$. Since (E_1, \dots, E_n) is an A_n -configuration, the geometric intersection numbers between the curves d_i satisfy

$$I(d_i, d_j) = \begin{cases} \frac{1}{2} & \text{if } |i-j| = 1, \\ 0 & \text{if } |i-j| \geq 2, \end{cases}$$

by Theorem 27. This shows that d_i intersects d_{i-1} and d_{i+1} only at its endpoints, and does not intersect any other curve d_j with $|i-j| > 1$. Such

a configuration of curves can be taken by the action of some $g \in \text{Diff}_0(\mathbb{C}; \Delta)$ to the standard configuration (c_1, \dots, c_n) so that $g(d_i) \simeq c_i$ for $i = 1, \dots, n$. Since $\pi_0(\text{Diff}_0(\mathbb{C}; \Delta)) \cong B_n$, this suffices to show that there is an element $\Psi \in \text{Br}(\mathcal{C})$ such that $\Psi(E_i) \cong \tilde{L}_i$. \square

7.4 The proof of the connectedness of $\text{Stab } \mathcal{C}$

Here we finish the proof of the connectedness of $\text{Stab } \mathcal{C}$. Let S_i be the simple $\mathcal{A}_n^{(1)}$ -module corresponding to the idempotent $(i) \in \mathcal{A}_n^{(1)}$ for $i = 1, \dots, n$, and \mathcal{A} be the full extension-closed subcategory of $\text{mod}_0 \mathcal{A}_n^{(1)}$ containing S_1, \dots, S_n . Then the standard t -structure of $D^b \text{mod } \mathcal{A}_n^{(1)}$ induces a bounded t -structure of \mathcal{C} with heart \mathcal{A} . For any connected component Σ of $\text{Stab } \mathcal{C}$, take a stability condition $\sigma \in \Sigma$ such that $Z(E) \notin \mathbb{R}$ for any spherical object $E \in \mathcal{C}$. Then σ gives an A_n -configuration (E_1, \dots, E_n) by Lemma 39. Applying Lemma 38 and Lemma 40, we can find an autoequivalence $\Psi \in \text{Br}(\mathcal{C})$ such that $\Psi(E_i) \cong S_i$. Then the heart of $\Psi\sigma$ coincides with \mathcal{A} and hence σ belongs to the distinguished connected component studied by Bridgeland in [11].

A Appendix

Let \mathcal{D} be the derived category $D^b \text{coh}_Z X$ of coherent sheaves on the minimal resolution X of an A_n -singularity supported at the exceptional set Z as in the body of this paper, and $\text{Auteq}^{FM} \mathcal{D}$ be the subgroup of $\text{Auteq } \mathcal{D}$ consisting of integral functors. A set of generators of $\text{Auteq}^{FM} \mathcal{D}$ is found in [30, theorem 1.3]. Here we prove that every autoequivalence of \mathcal{D} is given by an integral functor,

$$\text{Auteq}^{FM} \mathcal{D} = \text{Auteq } \mathcal{D}. \quad (8)$$

A.1 Weak ample sequence and Fourier-Mukai transform

Let \mathcal{A} be an abelian category over a field k . First we recall the definition of ample sequences in \mathcal{A} , introduced by Bondal and Orlov [38, §2], [5, appendix]:

Definition 41. A collection $\{P_i\}_{i \in \mathbb{Z}}$ of objects in \mathcal{A} is an *ample sequence* if for any object $F \in \mathcal{A}$, there is an integer N such that for any integer i smaller than N , the following conditions are satisfied:

- (i) the canonical morphism $\text{Hom}_{\mathcal{A}}(P_i, F) \otimes P_i \rightarrow F$ is surjective,
- (ii) $\text{Ext}_{\mathcal{A}}^j(P_i, F) = 0$ for any $j \neq 0$,
- (iii) $\text{Hom}_{\mathcal{A}}(F, P_i) = 0$.

When X is a projective variety, an ample line bundle L on X provides an example $\{L^{\otimes i}\}_{i \in \mathbb{Z}}$ of an ample sequence in $\text{coh } X$.

Let $\{P_i\}_{i \in \mathbb{Z}}$ be an ample sequence in \mathcal{A} , $D^b(\mathcal{A})$ be the bounded derived category of \mathcal{A} , and \mathcal{B} be the full subcategory of $D^b(\mathcal{A})$ consisting of $\{P_i\}_{i \in \mathbb{Z}}$. The inclusion functor will be denoted by $j : \mathcal{B} \hookrightarrow D^b(\mathcal{A})$. The following result is due to Bondal and Orlov:

Proposition 42. *Let Ψ be an autoequivalence of $D^b(\mathcal{A})$ and assume that there is an isomorphism of functors $g : j \xrightarrow{\sim} \Psi|_{\mathcal{B}}$. Then g can be extended to an isomorphism $\text{id} \xrightarrow{\sim} \Psi$ on the whole $D^b(\mathcal{A})$.*

The following lemma is proved in [30, claim 3.8.]:

Lemma 43. *Let $j' : \text{mod}_0 \mathcal{A}_n^{(1)} \hookrightarrow \mathcal{D}$ be the inclusion given by the McKay correspondence in §3. Then for any autoequivalence $\Phi \in \text{Auteq } \mathcal{D}$, there is an autoequivalence $\Psi \in \text{Auteq}^{FM} \mathcal{D}$ such that we have an isomorphism of functors $h : j' \xrightarrow{\sim} \Psi \circ \Phi|_{\text{mod}_0 \mathcal{A}_n^{(1)}}$.*

If $\text{mod}_0 \mathcal{A}_n^{(1)}$ has an ample sequence, then we can apply Proposition 42 to obtain the equality (8). But unfortunately, not all simple objects F of $\text{mod}_0 \mathcal{A}_n^{(1)}$ can satisfy the condition (iii). We rather consider $\text{coh}_Z X$. Although no sequence in $\text{coh}_Z X$ satisfies the conditions (i) and (ii) simultaneously for $F = \mathcal{O}_x$, we can weaken the condition (ii) in the following way:

Definition 44. A sequence $\{P_i\}_{i \in \mathbb{Z}}$ of objects in \mathcal{A} is a *weak ample sequence* if for any objects $F, G \in \mathcal{A}$, there is an integer N such that for any $i < N$, the condition (i) and (iii) in Definition 41 and the following condition (ii)' are satisfied;

(ii)' there are a natural number l and a surjection $\varphi : P_i^{\oplus l} \rightarrow G$ such that the pull-back

$$\varphi^* : \text{Ext}_{\mathcal{A}}^j(G, F) \rightarrow \text{Ext}_{\mathcal{A}}^j(P_i^{\oplus l}, F)$$

is the zero map for any $j \neq 0$.

The proof of Proposition 42 goes through also for weak ample sequences:

Proposition 45. *Let $j : \mathcal{B} \hookrightarrow D^b(\mathcal{A})$ be inclusion of the full subcategory consisting of a weak ample sequence of \mathcal{A} . Let Ψ be an autoequivalence of $D^b(\mathcal{A})$ and assume that there is an isomorphism of functors $g : j \xrightarrow{\sim} \Psi|_{\mathcal{B}}$. Then g can be extended to an isomorphism $\text{id} \xrightarrow{\sim} \Psi$ on the whole $D^b(\mathcal{A})$.*

A.2 A weak ample sequence in $\text{coh}_Z X$

Here we prove the equality (8). First we show that $\text{coh}_Z X$ has a weak ample sequence. Consider Z as the fundamental cycle with its scheme structure.

Lemma 46. *Let H be an ample divisor on X such that the divisor $H + Z$ is ample and put*

$$P_i := \mathcal{O}_{|i|Z}(iH)$$

for $i \in \mathbb{Z}$. Then $\{P_i\}_{i \in \mathbb{Z}}$ is a weak ample sequence in $\text{coh}_Z X$.

Proof. Let F and G be objects of $\text{coh}_Z X$. We show that there is an integer N such that the conditions (i), (ii)', and (iii) hold for any integer i smaller than N .

(i) Let N_1 be a negative integer such that F is an $\mathcal{O}_{-N_1 Z}$ -module, and N_2 be another integer such that the canonical morphism

$$\text{Hom}_{-N_1 Z}(\mathcal{O}_{-N_1 Z}(N_2 H), F) \otimes \mathcal{O}_{-N_1 Z}(N_2 H) \rightarrow F$$

is surjective. Then we can take $N = \min\{N_1, N_2\}$.

(iii) Let N be a negative integer such that

$$\text{Hom}_X(F, \mathcal{O}_Z(i(H + Z))) = 0 \tag{9}$$

for any $i < N$. The short exact sequences

$$0 \rightarrow \mathcal{O}_{(-i-j-1)Z}(iH - (j+1)Z) \rightarrow \mathcal{O}_{(-i-j)Z}(iH - jZ) \rightarrow \mathcal{O}_Z(iH - jZ) \rightarrow 0$$

for $j = 0, \dots, -i-2$ show that P_i has a filtration whose subquotients consist of $\mathcal{O}_Z(iH - jZ)$ for $j = 0, \dots, -i-1$. Since we have an inclusion

$$\mathcal{O}_Z(iH - jZ) \hookrightarrow \mathcal{O}_Z(i(H + Z))$$

for $j \leq -i$, (9) implies

$$\text{Hom}_X(F, \mathcal{O}_Z(iH - jZ)) = 0$$

and hence

$$\text{Hom}_X(F, P_i) = 0.$$

(ii)' Since $\text{Ext}_X^2(P_i, F) = 0$ for sufficiently small i by (iii) and the Serre duality, we only need to check the case $j = 1$. Take a negative integer N_1 such that the canonical morphism

$$\varphi : \text{Hom}_X(P_i, G) \otimes P_i \rightarrow G$$

is surjective for any $i < N_1$. We will prove that there is an integer $N < N_1$ such that for any $i < N$ and any extension

$$0 \rightarrow F \rightarrow E \rightarrow G \rightarrow 0 \tag{10}$$

of G by F corresponding to $e \in \text{Ext}_X^1(G, F)$, the extension

$$0 \rightarrow F \rightarrow E' \rightarrow \text{Hom}_X(P_i, G) \otimes P_i \rightarrow 0 \tag{11}$$

corresponding to

$$\varphi^*(e) = e \circ \varphi \in \text{Ext}_X^1(\text{Hom}_X(P_i, G) \otimes P_i, F)$$

splits.

First note that if one takes a sufficiently small i so that F and G are \mathcal{O}_{-iZ} -modules, then E is an \mathcal{O}_{-2iZ} -module for any $e \in \text{Ext}_X^1(G, F)$. Hence there is an integer $N_2 < N_1$ such that E , F , and G are \mathcal{O}_{-iZ} -modules for any $i < N_2$. Since H is ample, there is an integer $N < N_2$ such that $\text{Ext}_{-iZ}^1(P_i, F) = 0$ for any $i < N$. Take any $i < N$. Then (11) is an exact sequence not only of \mathcal{O}_X -modules but also of \mathcal{O}_{-iZ} -modules. Since $\text{Ext}_{-iZ}^1(P_i, F) = 0$, it splits as an exact sequence of \mathcal{O}_{-iZ} -modules, and hence as an exact sequence of \mathcal{O}_X -modules. \square

The above weak ample sequence of $\text{coh}_Z X$ belongs also to $\text{mod}_0 \mathcal{A}_n^{(1)}[-1]$:

Lemma 47. *If $i < 0$, then P_i in Lemma 46 belongs to ${}^{-1}\text{Per}(X/Y)[-1]$, which is identified with $\text{mod } \mathcal{A}_n^{(1)}[-1]$ by the McKay correspondence in §3.*

Proof. It follows from the definition of ${}^{-1}\text{Per}(X/Y)$ that a coherent sheaf F on X belongs to ${}^{-1}\text{Per}(X/Y)[-1]$ if and only if $f_*F = 0$. As in the proof of (iii) in Lemma 46, we have $f_*P_i = 0$ for $i < 0$ and hence $P_i \in {}^{-1}\text{Per}(X/Y)[-1]$. \square

Now we can prove the equality (8). Let Φ be an autoequivalence of \mathcal{D} . Then there is another autoequivalence $\Psi \in \text{Auteq}^{FM} \mathcal{D}$ such that we have an isomorphism of functors

$$h : j' \xrightarrow{\sim} \Psi \circ \Phi|_{\text{mod}_0 \mathcal{A}_n^{(1)}}$$

by Lemma 43. Let $j : \mathcal{B} \hookrightarrow \mathcal{D}$ be the inclusion of the full subcategory consisting of the weak ample sequence $\{P_i\}_{i < 0}$ given in Lemma 46. Since $\mathcal{B} \subset \text{mod}_0 \mathcal{A}_n^{(1)}[-1]$ by Lemma 47, h induces an isomorphism

$$g : j \xrightarrow{\sim} \Psi \circ \Phi|_{\mathcal{B}}$$

of functors from \mathcal{B} to \mathcal{D} . Since $\{P_i\}_{i \in \mathbb{Z}}$ is a weak ample sequence in $\text{coh}_Z X$ by Lemma 46 and $\mathcal{D} = D^b \text{coh}_Z X$, g can be extended to an isomorphism of functors on the whole \mathcal{D} by Proposition 45.

References

- [1] M. Artin & J.-L. Verdier, *Reflexive modules over rational double points*, Math. Ann., 270(1):79–82, 1985, MR 0769609, Zbl 0553.14001.
- [2] P. Aspinwall & M. Douglas, *D-brane stability and monodromy*, J. High Energy Phys., (5):no. 31, 35, 2002, MR 1915365.

- [3] A. Beilinson, J. Bernstein, & P. Deligne, *Faisceaux pervers, Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982, MR 0751966, Zbl 0536.14011.
- [4] A. Bondal & M. Kapranov, *Enhanced triangulated categories*, *Mat. Sb.*, 181(5):669–683, 1990, MR 1055981, Zbl 0729.18008.
- [5] A. Bondal & D. Orlov, *Reconstruction of a variety from the derived category and groups of autoequivalences*, *Compositio Math.*, 125(3):327–344, 2001, MR 1818984, Zbl 0994.18007.
- [6] J. Bondorf, *A Fourier transformation for Higgs bundles*, *J. Reine Angew. Math.*, 591:21–48, 2006, MR 2212878, Zbl 1094.14023.
- [7] T. Bridgeland, *Fourier-Mukai transforms for elliptic surfaces*, *J. Reine Angew. Math.*, 498:115–133, 1998, MR 1629929, Zbl 0905.14020.
- [8] T. Bridgeland, *Flops and derived categories*, *Invent. Math.*, 147(3):613–632, 2002, MR 1893007, Zbl 1085.14017.
- [9] T. Bridgeland, *Stability conditions on triangulated categories*, *Ann. of Math.*, 166(2):317–345, 2007, MR 2373143, Zbl 1137.18008.
- [10] T. Bridgeland, *Stability conditions on K3 surfaces*, *Duke Math. J.* 141, No. 2, 241–291, 2008, MR 2376815, Zbl 1138.14022.
- [11] T. Bridgeland, *Stability conditions and Kleinian singularities*, *Intern. Math. Res. Notices* 21:4142–4157, 2009.
- [12] T. Bridgeland, A. King, & M. Reid, *The McKay correspondence as an equivalence of derived categories*, *J. Amer. Math. Soc.*, 14(3):535–554 (electronic), 2001, MR 1824990, Zbl 0966.14028.
- [13] E. Brieskorn, *Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe*, *Invent. Math.*, 12:57–61, 1971, MR 0293615, Zbl 0204.56502.
- [14] P. Deligne, *Les immeubles des groupes de tresses généralisés*, *Invent. Math.*, 17:273–302, 1972, MR 0422673, Zbl 0238.20034.
- [15] M. Douglas, *D-branes, categories and $N = 1$ supersymmetry*, *J. Math. Phys.*, 42(7):2818–2843, 2001, *Strings, branes, and M-theory*, MR 1840318, Zbl 1036.81027.
- [16] M. Douglas, *D-branes on Calabi-Yau manifolds*, *European Congress of Mathematics, Vol. II (Barcelona, 2000)*, volume 202 of *Progr. Math.*, pages 449–466. Birkhäuser, Basel, 2001, MR 1909947, Zbl 1027.81030.

- [17] M. Douglas, *Dirichlet branes, homological mirror symmetry, and stability*, *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pages 395–408, Beijing, 2002. Higher Ed. Press, MR 1957548, Zbl 1008.81074.
- [18] M. Douglas, B. Fiol, & C. Römelsberger, *Stability and BPS branes*, *J. High Energy Phys.*, no. 9, 006, 15 pp. (electronic), 2005, MR 2171358.
- [19] Nguyễn Việt Dũng, *The fundamental groups of the spaces of regular orbits of the affine Weyl groups*, *Topology*, 22(4):425–435, 1983, MR 0715248, Zbl 0524.57015.
- [20] H. Esnault, *Reflexive modules on quotient surface singularities*, *J. Reine Angew. Math.*, 362:63–71, 1985, MR 0809966, Zbl 0553.14016.
- [21] A. Floer, *Morse theory for Lagrangian intersections*, *J. Differential Geom.*, 28(3):513–547, 1988, MR 0965228, Zbl 0674.57027.
- [22] K. Fukaya, *Morse homotopy, A^∞ -category, and Floer homologies*, *Proceedings of GARC Workshop on Geometry and Topology '93 (Seoul, 1993)*, volume 18 of *Lecture Notes Ser.*, pages 1–102, Seoul, 1993. Seoul Nat. Univ, MR 1270931, Zbl 0853.57030.
- [23] K. Fukaya, *Floer homology and mirror symmetry II, Minimal surfaces, geometric analysis and symplectic geometry (Baltimore, MD, 1999)*, volume 34 of *Adv. Stud. Pure Math.*, pages 31–127. Math. Soc. Japan, Tokyo, 2002, MR 1925734, Zbl 1030.53087.
- [24] K. Fukaya, Y.-G. Oh, H. Ohta, & K. Ono, *Lagrangian intersection Floer theory*, preprint available at <http://www.math.kyoto-u.ac.jp/~fukaya/>.
- [25] A. Grothendieck, *Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II*, *Inst. Hautes Études Sci. Publ. Math.*, No. 17, 1963, MR 0163911, Zbl 0122.16102.
- [26] J. Hashiba & M. Naka, *Landau-Ginzburg description of D-branes on ALE spaces*, *Nuclear Phys. B*, 599(1-2):361–380, 2001, MR 1826465, Zbl 1097.81718.
- [27] K. Hori, A. Iqbal, & C. Vafa, *D-branes and mirror symmetry*, arXiv:hep-th/0005247v2.
- [28] K. Hori & Cumrun Vafa, *Mirror symmetry*, arXiv:hep-th/0002222v3.
- [29] M. Inaba, *Toward a definition of moduli of complexes of coherent sheaves on a projective scheme*, *J. Math. Kyoto Univ.*, 42(2):317–329, 2002, MR 1966840, Zbl 1063.14013.

- [30] A. Ishii & H. Uehara, *Autoequivalences of derived categories on the minimal resolutions of A_n -singularities on surfaces*, J. Differential Geom., 71(3):385–435, 2005, MR 2198807, Zbl 1097.14013.
- [31] M. Kapranov & E. Vasserot, *Kleinian singularities, derived categories and Hall algebras*, Math. Ann., 316(3):565–576, 2000, MR 1752785, Zbl 0997.14001.
- [32] R. Kent, IV & D. Peifer, *A geometric and algebraic description of annular braid groups*, Internat. J. Algebra Comput., 12(1-2):85–97, 2002, International Conference on Geometric and Combinatorial Methods in Group Theory and Semigroup Theory (Lincoln, NE, 2000), MR 1902362, Zbl 1010.20024.
- [33] M. Khovanov & P. Seidel, *Quivers, Floer cohomology, and braid group actions*, J. Amer. Math. Soc., 15(1):203–271 (electronic), 2002, MR 1862802, Zbl 1035.53122.
- [34] M. Kontsevich, *Homological algebra of mirror symmetry, Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 120–139, Basel, 1995. Birkhäuser, MR 1403918, Zbl 0846.53021.
- [35] J. Moser, *On the volume elements on a manifold*, Trans. Amer. Math. Soc., 120:286–294, 1965, MR 0182927, Zbl 0141.19407.
- [36] D. Mumford, *Projective invariants of projective structures and applications, Proc. Internat. Congr. Mathematicians (Stockholm, 1962)*, pages 526–530. Inst. Mittag-Leffler, Djursholm, 1963, MR 0175899, Zbl 0154.20702.
- [37] S. Okada, *On stability manifolds of Calabi-Yau surfaces*, Int. Math. Res. Not., Art.ID 58743, 16 pp., 2006, MR 2276354, Zbl 1122.14015.
- [38] D. Orlov, *Equivalences of derived categories and K3 surfaces*, J. Math. Sci. (New York), 84(5):1361–1381, 1997, Algebraic geometry, 7, MR 1465519, Zbl 0938.14019.
- [39] P. Seidel, *Homological mirror symmetry for the quartic surface*, math.AG/0310414.
- [40] P. Seidel, *Graded Lagrangian submanifolds*, Bull. Soc. Math. France, 128(1):103–149, 2000, MR 1765826, Zbl 0992.53059.
- [41] P. Seidel & R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J., 108(1):37–108, 2001, MR 1831820, Zbl 1092.14025.
- [42] R. Thomas, *Stability conditions and the braid group*, Comm. Anal. Geom., 14(1):135–161, 2006, MR 2464932, Zbl pre05135052.

- [43] M. Van den Bergh, *Three-dimensional flops and noncommutative rings*,
Duke Math. J., 122(3):423–455, 2004, MR 2057015, Zbl 1074.14013.

Akira Ishii

Department of Mathematics, Graduate School of Science, Hiroshima
University, 1-3-1 Kagamiyama, Higashi-Hiroshima, 739-8526, Japan
e-mail address : akira@math.sci.hiroshima-u.ac.jp

Kazushi Ueda

Department of Mathematics, Graduate School of Science, Osaka Univer-
sity, Machikaneyama 1-1, Toyonaka, Osaka, 560-0043, Japan.
e-mail address : kazushi@math.sci.osaka-u.ac.jp

Hokuto Uehara

Department of Mathematics and Information Sciences, Tokyo Metropoli-
tan University, 1-1 Minami-Ohsawa, Hachioji-shi, Tokyo, 192-0397, Japan
e-mail address : hokuto@tmu.ac.jp